PERIYAR UNIVERSITY

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CENTRE FOR DISTANCE AND ONLINE EDUCATION
(CDOE)

MASTER OF SCIENCE IN MATHEMATICS
SEMESTER - I

ELECTIVE COURSE: ORDINARY DIFFERENTIAL EQUATIONS
(Candidates admitted from 2024 onwards)

PERIYAR UNIVERSITY

CENTRE FOR DISTANCE AND ONLINE EDUCATION (CDOE)

M.Sc., MATHEMATICS 2024 admission onwards

ELECTIVE
Ordinary Differential Equations

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TEXTBOOK:
E. A. Coddington, An Introduction to Ordinary Differential Equations, Prentice-Hall of India Ltd., New Delhi, 2011.

Unit 1

Linear Equations with Constant Coefficients

OBJECTIVE:
After the successful completion of this unit, the students are expected to recall the basic concept of linear homogeneous and non-homogeneous differential equations with constant coefficients. Also, the solution of initial value problems for second-order equations. In particular, we study linear independence and dependence results using the Wronskian formula.

1.1 Introduction

In this unit, you will learn about the basics of linear equations with constant coefficients and the second order homogeneous equations. A differential equation is an equation which contains derivatives of one or more depended variables with respect to one or more independent variables.

A linear differential equation with constant coefficients of order n has the form

$$a_0 y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = b(x), \tag{1.1}$$

where $a_0, a_1, a_2, \cdots, a_n$ are complex constants with $a_0 \neq 0$, and b is a complex-valued function on an interval I.

By dividing 1.1 by a_0 and assuming $a_0 = 1$, the equation 1.1 becomes

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = b(x). \tag{1.2}$$

It will be more convenient to denote the differential expression on the left side of the equation 1.2 as $L(y)$. Thus

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y, \tag{1.3}$$

and the equation 1.2 becomes simply $L(y) = b(x)$.

Definition 1.1 *If $b(x) = 0$ for all $x \in I$, then the corresponding equation $L(y) = 0$ is called a homogeneous equation, whereas if $b(x) \neq 0$ for some $x \in I$, then the equation $L(y) = b(x)$ is called a non-homogeneous equation.*

Definition 1.2 *We denote L is a differential operator which operates on a function ϕ which have n derivatives on I, and the value of a function $L(\phi)$ at x is given by*

$$L(\phi)(x) = \phi^{(n)}(x) + a_1\phi^{(n-1)}(x) + \cdots + a_n\phi(x).$$

As a result, we get

$$L(\phi) = \phi^{(n)} + a_1\phi^{(n-1)} + \cdots + a_n\phi.$$

Definition 1.3 *A solution of $L(y) = b(x)$ is a function ϕ with n derivatives on I that satisfy $L(\phi) = b$.*

Remark 1.1 *If b is continuous on I, then it is possible to find all solutions of $L(y) = b(x)$.*

1.2 The second order homogeneous equation

First we consider the first order equation with constant coefficients

$$y' + ay = 0, \tag{1.4}$$

where a is a complex constant. Assume that ϕ is a solution of 1.4. Then

$$\begin{aligned} \phi' + a\phi &= 0 \\ \implies e^{ax}(\phi' + a\phi) &= 0 \\ \implies (e^{ax}\phi)' &= 0. \end{aligned}$$

Therefore $e^{ax}\phi(x) = c$, for some constant c. Hence

$$\phi(x) = ce^{-ax}.$$

The constant $-a$ in the above solution is the solution of the equation $r + a = 0$. We have seen that the above method works for equation of the first order. Let us try it for the second order homogeneous equation.

Theorem 1.1 *Consider the equation*

$$L(y) = y'' + a_1y' + a_2y = 0, \tag{1.5}$$

where a_1 and a_2 are constants. If r_1 and r_2 are distinct roots of the characteristic polynomial p, where

$$p(r) = r^2 + a_1r + a_2,$$

then the functions ϕ_1 and ϕ_2 defined by

$$\phi_1(x) = e^{r_1x}, \ \ \phi_2(x) = e^{r_2x} \tag{1.6}$$

are solutions of $L(y) = 0$. If r_1 is a repeated root of p, then the functions ϕ_1 and ϕ_2 defined by

$$\phi_1(x) = e^{r_1x}, \ \ \phi_2(x) = xe^{r_1x} \tag{1.7}$$

are solutions of $L(y) = 0$.

Proof:
Let $y = e^{rx}$ is a solution of $L(y)$, where r is a constant. Then $y' = re^{rx}, y'' = r^2e^{rx}$. Then 1.5 becomes

$$\begin{aligned} L(e^{rx}) &= r^2e^{rx} + a_1re^{rx} + a_2e^{rx} = 0 \\ &\Longrightarrow (r^2 + a_1r + a_2)e^{rx} = 0 \\ &\Longrightarrow p(r)e^{rx} = 0 \\ &\Longleftrightarrow p(r) = 0 \quad \because e^{rx} \neq 0 \end{aligned} \tag{1.8}$$

Thus e^{rx} is a solution of $L(y)$ iff r is a root of the characteristic polynomial $p(r)$.
Since $p(r) = r^2 + a_1r + a_2$ is a polynomial of degree two, it has two complex roots, namely, r_1 and r_2 (by the fundamental theorem of algebra). We have the following two cases:
Case 1: Distinct roots ($r_1 \neq r_2$)
If r_1 and r_2 are two distinct solutions of $p(r)$, then e^{r_1x} and e^{r_2x} are two distinct solutions of $L(y) = 0$.
Case 2: Repeated roots ($r_1 = r_2$)
We have

$$L(e^{rx}) = p(r)e^{rx}, \tag{1.9}$$

for all r and x. We recall that if r_1 is a repeated root of $p(r)$, then $p(r_1) = 0$ and $p'(r_1) = 0$. Differentiating 1.9 with respect to r will give us

$$\begin{aligned} L\left(\frac{\partial}{\partial r}e^{rx}\right) &= \frac{\partial}{\partial r}(p(r)e^{rx}) \\ L\left(xe^{rx}\right) &= p'(r)e^{rx} + p(r)xe^{rx} \\ &= [p'(r) + xp(r)]e^{rx}. \end{aligned}$$

Now substituting $r = r_1$ in this equation we get $L(xe^{r_1x}) = 0$, thus xe^{r_1x} is another solution in case $r_1 = r_2$.
Result:
If ϕ_1, ϕ_2 are any two solutions of $L(y) = 0$ and c_1, c_2 are the two constants ,then the function $\phi = c_1\phi_1 + c_2\phi_2$ is also a solution of $L(y) = 0$.
Proof:

$$\begin{aligned} L(\phi) &= (c_1\phi_1 + c_2\phi_2)'' + a_1(c_1\phi_1 + c_2\phi_2)' + a_2(c_1\phi_1 + c_2\phi_2) \\ &= c_1\phi_1'' + c_2\phi_2'' + a_1c_1\phi_1' + a_1c_2\phi_2' + a_2c_1\phi_1 + a_2c_2\phi_2 \\ &= c_1(\phi_1'' + a_1\phi_1' + a_2\phi_1) + c_2(\phi_2'' + a_1\phi_2' + a_2\phi_2) \\ &= c_1L(\phi_1) + c_2L(\phi_2) \\ &= c_1(0) + c_2(0) \\ &= 0. \end{aligned}$$

Example 1.1 *Find all the solutions of the equation $y'' + y' - 2y = 0$.*

Solution:
Consider the equation $y''+y'-2y = 0$. The characteristic polynomial is $p(r) = r^2+r-2$. Let $p(r) = 0$. Then

$$\Longrightarrow r^2 + r - 2 = 0 \Rightarrow r = -2, 1.$$

The roots are $-2, 1$. Therefore every solution ϕ has the form $\phi(x) = c_1e^{-2x} + c_2e^x$, where $c_1$and c_2 are constants.

Example 1.2 *Find all the solutions of the equation $y'' + \omega^2 y = 0$.*

Solution:
The characteristic polynomial of the given equation $y'' + \omega^2 y = 0$ is

$$p(r) = r^2 + \omega^2.$$

Assuming $p(r) = 0$, we find that the roots of this polynomial are $i\omega$ and $-i\omega$. Thus, every solution ϕ takes the form $\phi(x) = c_1e^{i\omega x} + c_2e^{-i\omega x}$, where $c_1$and c_2 are any two constants.
Note:

(i) Taking $c_1 = \frac{1}{2}, c_2 = \frac{1}{2}$, we see that $\cos \omega x$ is a solution.

(ii) Taking $c_1 = \frac{1}{2}i, c_2 = -\frac{1}{2}i$, we see that $\sin \omega x$ is a solution.

(iii) The equation $y'' + \omega^2 y = 0$ is known as the harmonic oscillator equation and is used to examine oscillatory behaviour in a variety of physical contexts.

Let us sum up

1. We have introduced the second order homogeneous equation.
2. We have discussed the roots of the second order homogeneous equation.
3. We have discussed the distinct and repeated roots of the characteristic polynomial p and it's solutions.
4. Finally, we solved some illustrative examples.

Check your progress

1. The equation $y'' + \sin y = 0$, $y(0) = y(2\pi)$; is
   (a) linear (b) linear homogeneous
   (c) linear nonhomogeneous (d) nonlinear
2. The harmonic oscillator equation is
   (a) $y'' + \omega^2 y = 0$ (b) $y'' - \omega y = 0$
   (c) $y'' - \omega^2 y = 0$ (d) $y'' - \omega^3 y = 0$

1.3 Initial value problems for second order equations

The show that every solution of the equation

$$L(y) = y'' + a_1y' + a_2y = 0. \tag{1.10}$$

is a linear combination of the solutions 1.6 or 1.7 will depend on proving that the initial value problems for this equation have unique solutions.

Definition 1.4 *An initial value problem of $L(y) = 0$ is a problem of finding a solution ϕ satisfying*

$$\phi(x_0) = \alpha, \qquad \phi'(x_0) = \beta, \tag{1.11}$$

where x_0 is some real number, and α, β are given constants. Thus we specify ϕ and its first derivative at some initial point x_0. This problem is denoted by

$$L(y) = 0, \qquad y(x_0) = \alpha, \qquad y'(x_0) = \beta. \tag{1.12}$$

Theorem 1.2 *(Existence Theorem)*
For any real x_0, and constants α, β, there exists a solution ϕ of the initial value problem 1.12 on $-\infty < x < \infty$.

Proof:
We prove that there are unique constants c_1, c_2 such that $\phi = c_1\phi_1 + c_2\phi_2$ satisfies 1.11, where ϕ_1, ϕ_2 are the solutions given by 1.6 or 1.7. In order to satisfy the relations 1.11 we must have

$$c_1\phi_1(x_0) + c_2\phi_2(x_0) = \alpha \tag{1.13}$$
$$c_1\phi_1'(x_0) + c_2\phi_2'(x_0) = \beta \tag{1.14}$$

and these equations will have a unique solution c_1, c_2 if the determinant

$$\Delta = \begin{vmatrix} \phi_1(x_0) & \phi_2(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) \end{vmatrix} = \phi_1(x_0)\phi_2'(x_0) - \phi_1'(x_0)\phi_2(x_0) \neq 0.$$

If $r_1 \neq r_2$, then

$$\phi_1(x) = e^{r_1 x}, \phi_2(x) = e^{r_2 x}, \phi_1'(x_0) = r_1 e^{r_1 x_0}, \phi_2'(x_0) = r_2 e^{r_2 x_0}$$

and

$$\begin{aligned} \Delta &= e^{r_1 x_0} r_2 e^{r_2 x_0} - r_1 e^{r_1 x_0} e^{r_2 x_0} \\ &= r_2 e^{r_1 x_0} e^{r_2 x_0} - r_1 e^{r_1 x_0} e^{r_2 x_0} \\ &= (r_2 - r_1) e^{(r_1 + r_2) x_0}, \end{aligned}$$

which is not zero, since $e^{(r_1+r_2)x_0} \neq 0$.
If $r_1 = r_2$, then

$$\phi_1(x) = e^{r_1 x}, \phi_2(x) = x e^{r_1 x}, \phi_1'(x_0) = r_1 e^{r_1 x_0}, \phi_2'(x_0) = x_0 r_1 e^{r_1 x_0} + e^{r_1 x_0}$$

and

$$\begin{aligned} \Delta &= (e^{r_1 x_0})(x_0 r_1 e^{r_1 x_0} + e^{r_1 x_0}) - (r_1 e^{r_1 x_0})(x_0 e^{r_1 x_0}) \\ &= e^{r_1 x_0} x_0 r_1 e^{r_1 x_0} + e^{r_1 x_0} e^{r_1 x_0} - r_1 e^{r_1 x_0} x_0 e^{r_1 x_0} \\ &= e^{r_1 x_0} x_0 r_1 e^{r_1 x_0} + e^{2 r_1 x_0} - r_1 x_0 e^{2 r_1 x_0} \\ &= e^{r_1 x_0}[e^{r_1 x_0} + x_0 r_1 e^{r_1 x_0} - r_1 x_0 e^{r_1 x_0}] \end{aligned}$$

$$= e^{2r_1x_0}$$
$$\neq 0.$$

As a result, the determinant condition is satisfied in both cases. Thus, if c_1,c_2 are the unique constants satisfying 1.13 and 1.14, the function

$$\phi = c_1\phi_1 + c_2\phi_2$$

will be the desired solution satisfying 1.11.

Note:

If b and c are any two constants, then

$$\begin{aligned} 0 &\leq (|b|-|c|)^2 = |b|^2 + |c|^2 - 2|b||c| \\ \Longrightarrow 2|b||c| &\leq |b|^2 + |c|^2. \end{aligned} \tag{1.15}$$

Theorem 1.3 *Let ϕ be any solution of $L(y) = y'' + a_1y' + a_2y = 0$ on an interval I containing a point x_0. Then for all x in I*

$$\|\phi(x_0)\|e^{-k|x-x_0|} \leq \|\phi(x)\| \leq \|\phi(x_0)\|e^{k|x-x_0|}, \tag{1.16}$$

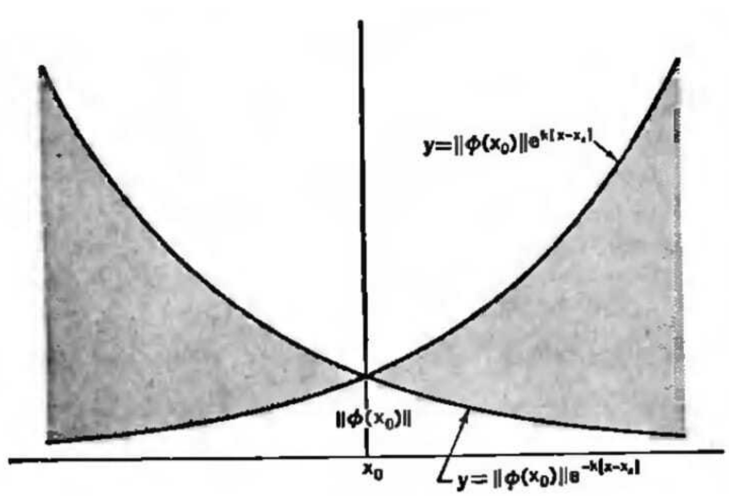
where $\|\phi(x)\| = [|\phi(x)|^2 + |\phi'(x)|^2]^{1/2}$, $k = 1 + |a_1| + |a_2|$.

Proof:

Let

$$u(x) = \|\phi(x)\|^2 \tag{1.17}$$

$$\begin{aligned} u(x) &= |\phi(x)|^2 + |\phi'(x)|^2 \\ u &= \phi\bar{\phi} + \phi'\bar{\phi}' \\ u' &= |\phi'\bar{\phi} + \phi\bar{\phi}' + \phi''\bar{\phi}' + \phi'\bar{\phi}''| \\ |u'(x)| &= |\phi'(x)\bar{\phi}(x) + \phi(x)\bar{\phi}'(x) + \phi''(x)\bar{\phi}'(x) + \phi'(x)\bar{\phi}''(x)| \\ &\leq |\phi'(x)||\bar{\phi}(x)| + |\phi(x)||\bar{\phi}'(x)| + |\phi''(x)||\bar{\phi}'(x)| + |\phi'(x)||\bar{\phi}''(x)| \\ &\leq |\phi'(x)||\phi(x)| + |\phi(x)||\phi'(x)| + |\phi''(x)||\phi'(x)| + |\phi'(x)||\phi''(x)| \\ &\leq 2|\phi'(x)||\phi(x)| + 2|\phi'(x)||\phi''(x)|. \end{aligned} \tag{1.18}$$



Since ϕ satisfies $L(\phi) = 0$ we get $\phi'' + a_1\phi' + a_2\phi = 0$.
Hence

$$|\phi''(x)| \leq |a_1||\phi'(x)| + |a_2||\phi(x)|. \tag{1.19}$$

Using 1.19 in 1.18 we have

$$\begin{aligned}
|u'(x)| &\leq 2|\phi(x)||\phi'(x)| + 2|\phi'(x)|[|a_1||\phi'(x)| + |a_2||\phi(x)|] \\
&\leq 2|\phi(x)||\phi'(x)| + 2|\phi'(x)||a_1||\phi'(x)| + 2|\phi'(x)||a_2||\phi(x)| \\
&\leq 2|\phi(x)||\phi'(x)| + 2|a_1||\phi'(x)|^2 + 2|\phi'(x)||a_2||\phi(x)|
\end{aligned}$$

$$\begin{aligned}
&\leq 2|\phi(x)||\phi'(x)| + 2|\phi'(x)||a_2||\phi(x)| + 2|a_1||\phi'(x)|^2 \\
&\leq 2|\phi(x)||\phi'(x)|[\ 1 + |a_2|\ ] + 2|a_1||\phi'(x)|^2 \\
&\leq 2(1 + |a_2|)|\phi(x)||\phi'(x)| + 2|a_1||\phi'(x)|^2.
\end{aligned} \tag{1.20}$$

Take $b = \phi(x)$ and $c = \phi'(x)$ in 1.15 we get the following inequality

$$2|\phi(x)||\phi'(x)| \leq |\phi(x)|^2 + |\phi'(x)|^2.$$

Equation 1.20 becomes

$$\begin{aligned}
|u'(x)| &\leq 1 + |a_2|(|\phi(x)|^2 + |\phi'(x)|^2) + 2|a_1||\phi'(x)|^2 \\
&\leq (1 + |a_2|)|\phi(x)|^2 + (1 + |a_2|)|\phi'(x)|^2 + 2|a_1||\phi'(x)|^2 \\
&\leq (1 + |a_2|)|\phi(x)|^2 + |\phi'(x)|^2[1 + |a_2| + 2|a_1|] \\
&\leq 2(1 + |a_1| + |a_2|)[|\phi(x)|^2 + |\phi'(x)|^2] \\
&\leq 2(1 + |a_1| + |a_2|)\|\phi(x)\|^2 \\
|u'(x)| &\leq 2ku(x).
\end{aligned}$$

This is equivalent to

$$-2ku(x) \leq u'(x) \leq 2ku(x). \tag{1.21}$$

Now consider the right inequality of 1.21

$$\begin{aligned}
u'(x) &\leq 2ku(x) \\
\implies u' - 2ku &\leq 0.
\end{aligned}$$

Then it is equivalent to

$$e^{-2ku}(u' - 2ku) = (e^{-2ku}u)' \leq 0.$$

Suppose that $x > x_0$. If integrate from x_0 to x, then we get

$$\begin{aligned}
e^{-2kx}u(x) - e^{-2kx_0}u(x_0) &\leq 0 \\
e^{-2kx}u(x) &\leq e^{-2kx_0}u(x_0) \\
u(x) &\leq u(x_0)e^{2k(x-x_0)} \\
\|\phi(x)\|^2 &\leq \|\phi(x_0)\|^2 e^{2k(x-x_0)}
\end{aligned}$$

$$\|\phi(x)\| \leq \|\phi(x_0)\|e^{k(x-x_0)} \text{ for } x > x_0. \tag{1.22}$$

Similarly the left inequality of 1.21 will give us

$$-2ku(x) \leq u'(x)$$
$$\|\phi(x_0)\|e^{-k(x-x_0)} \leq \|\phi(x)\| \text{ for } x > x_0. \tag{1.23}$$

Now, combining the inequalities 1.22 and 1.23 we get

$$\|\phi(x_0)\|e^{-k(x-x_0)} \leq \|\phi(x)\| \leq \|\phi(x_0)\|e^{k(x-x_0)} \text{ for } x > x_0. \tag{1.24}$$

A consideration of 1.21 for the case $x < x_0$ together with an integration from x to x_0 yields

$$\|\phi(x_0)\|e^{k(x-x_0)} \leq \|\phi(x)\| \leq \|\phi(x_0)\|e^{-k(x-x_0)} \text{ for } x < x_0. \tag{1.25}$$

Now, combining the inequalities 1.24 and 1.25 we get

$$\|\phi(x_0)\|e^{-k|x-x_0|} \leq \|\phi(x)\| \leq \|\phi(x_0)\|e^{k|x-x_0|}.$$

Remark:
Geometrically the inequality 1.16 says that $\|\phi(x)\|$ always remains between the two curves $y = \|\phi(x_0)\|e^{k|x-x_0|}$ and $y = \|\phi(x_0)\|e^{-k|x-x_0|}$.

Theorem 1.4 *(Uniqueness Theorem)*
Let α, β be any two constants, and let x_0 be any real number. On any interval I containing x_0 there exists at most one solution ϕ of the initial value problem

$$L(y) = 0, \quad y(x_0) = \alpha, \quad y'(x_0) = \beta.$$

Proof:
Suppose that ϕ and ψ are two solutions of $L(y) = 0$. Then

$$L(\phi) = 0, \phi(x_0) = \alpha, \phi'(x_0) = \beta \quad \text{and}$$
$$L(\psi) = 0, \psi(x_0) = \alpha, \psi'(x_0) = \beta.$$

Let $\chi = \phi - \psi$. Then

$$L(\chi) = L(\phi) - L(\psi) = 0 \text{ and } \chi(x_0) = 0, \chi'(x_0) = 0.$$

Therefore $\|\chi(x_0)\| = 0$. By the existence theorem we have

$$\|\chi(x_0)\|e^{-k|x-x_0|} \leq \|\chi(x)\| \leq \|\chi(x_0)\|e^{k|x-x_0|}. \tag{1.26}$$

Thus $\|\chi(x)\| = 0, \ \forall x \in I$. This implies $\chi(x) = 0, \ \forall x \in I$. Since $\chi = \phi - \psi$, we obtain

$$\phi(x) - \psi(x) = 0, \quad \forall x \in I,$$
$$\phi(x) = \psi(x), \quad \forall x \in I,$$
$$\implies \phi = \psi.$$

Hence proved the uniqueness theorem.

Theorem 1.5 *Let ϕ_1, ϕ_2 be the two solutions of $L(y) = 0$ given by 1.6 in case $r_1 \neq r_2$, and by 1.7 in case $r_1 = r_2$. If c_1, c_2 are any two constants, then the function $\phi = c_1\phi_1 + c_2\phi_2$ is a solution of $L(y) = 0$ on $-\infty < x < \infty$. Conversely, if ϕ is any solution of $L(y) = 0$ on $-\infty < x < \infty$, then there are unique constants c_1, c_2 such that $\phi = c_1\phi_1 + c_2\phi_2$.*

Proof:
First we prove $\phi = c_1\phi_1 + c_2\phi_2$ is a solution of $L(y)$.
Given ϕ_1, ϕ_2 are the solutions of $L(y) = 0$. Then $L(\phi_1) = 0$ and $L(\phi_2) = 0$.
Since $L(y) = y'' + a_1y' + a_2y$ and $\phi = c_1\phi_1 + c_2\phi_2$, we get

$$\begin{aligned}
L(\phi) &= \phi'' + a_1\phi' + a_2\phi \\
&= (c_1\phi_1 + c_2\phi_2)'' + a_1(c_1\phi_1 + c_2\phi_2)' + a_2(c_1\phi_1 + c_2\phi_2) \\
&= c_1\phi_1'' + c_2\phi_2'' + a_1c_1\phi_1' + a_1c_2\phi_2' + a_2c_1\phi_1 + a_2c_2\phi_2 \\
&= c_1[\phi_1'' + a_1\phi_1' + a_2\phi_1] + c_2[\phi_2'' + a_1\phi_2' + a_2\phi_2] \\
&= c_1L(\phi_1) + c_2L(\phi_2) \\
&= 0 \qquad\qquad \because L(\phi_1) = L(\phi_2) = 0.
\end{aligned}$$

Hence the function ϕ is a solution of $L(y) = 0$.

Conversely, assume that $\phi = c_1\phi_1 + c_2\phi_2$ is a solution of $L(y) = 0$. Let x_0 be any real number and α, β be two given constants. In the proof of existence theorem we showed that there is a solution ψ of $L(y) = 0$ satisfying $\psi(x_0) = \alpha, \psi'(x_0) = \beta$ of the form

$$\psi = c_1\phi_1 + c_2\phi_2,$$

where c_1, c_2 are uniquely determined by α, β. By uniqueness theorem $\phi = \psi$. Hence the proof.

Let us sum up

1. We have discussed the existence and uniqueness theorem of the initial value problem.

2. We have proved the two solutions of $L(y) = 0$, then the linear combination of those two solutions is also a solution of $L(y) = 0$.

3. Finally, we solved some illustrative examples.

Check your progress

3. Consider the initial value problem, $y' = y^2, y(0) = 1, (x, y) \in \mathbb{R} \times \mathbb{R}$. Then there exists a unique solution of the IVP on
   (a) (-∞, ∞) (b) ($-\infty$, 1) (c) (-2, 2) (d) (-1,∞)

4. The solution of the differential equation $5y'' + 3y' = 0$ is given by
   (a) $y = c_1e^{3x} + c_2e^{5x}$ (b) $y = c_1 + c_2e^{-\frac{3}{5}x}$
   (c) $y = (c_1 + c_2x)e^{5x}$ (d) $y = c_1e^{3ix} + c_2e^{-3ix}$

5. State the existence theorem for solutions of a second order initial value problem, with constant coefficients.

6. State the uniqueness theorem for solutions of a second order initial value problem, with constant coefficients.

1.4 Linearly dependence and independence

Definition 1.5 *Two functions ϕ_1, ϕ_2 defined on an interval I are said to be linearly dependent on I, if there exist two constants c_1, c_2, not both zero, such that*

$$c_1\phi_1(x) + c_2\phi_2(x) = 0, \quad \forall\, x \in I.$$

Definition 1.6 *The functions ϕ_1, ϕ_2 are said to be linearly independent on I, if they are not linearly dependent there. That is, if the only constants c_1, c_2 such that $c_1\phi_1(x) + c_2\phi_2(x) = 0, \ \forall\, x \in I$ are the constants $c_1 = 0, c_2 = 0$.*

Example 1.3 *The functions $\phi_1(x) = e^{r_1 x}, \phi_2(x) = e^{r_2 x}$ are linearly independent on any interval I.*

Solution:
Suppose $c_1\phi_1(x) + c_2\phi_2(x) = 0, \ \forall\, x \in I$. Then

$$c_1 e^{r_1 x} + c_2 e^{r_2 x} = 0, \ \forall\, x \in I. \tag{1.27}$$

Multiplying the above equation by $e^{-r_1 x}$ we get

$$\begin{aligned} c_1 e^0 + c_2 e^{(r_2 - r_1)} x &= 0 \\ \implies \ c_1 + c_2 e^{(r_2 - r_1)x} &= 0. \end{aligned} \tag{1.28}$$

Differentiating the last equation with respect to x we get

$$c_2(r_2 - r_1)e^{(r_2 - r_1)x} = 0.$$

Since $r_2 - r_1 \neq 0$ and $e^{(r_2 - r_1)x} \neq 0$ we have $c_2 = 0$. Substituting $c_2 = 0$ in the equation 1.28 will give us $c_1 = 0$.

Hence ϕ_1 and ϕ_2 are linearly independent.

Example 1.4 *The functions $\phi_1 = e^{r_1 x}, \phi_2(x) = x e^{r_1 x}$ are linearly independent on any interval I.*

Solution:
Suppose $c_1\phi_1(x) + c_2\phi_2(x) = 0$. Then

$$c_1 e^{r_1 x} + c_2 x e^{r_1 x} = 0. \tag{1.29}$$

Multiplying the last equation by $e^{-r_1 x}$ we get

$$\begin{aligned} c_1 e^0 + c_2 e^0 x &= 0 \\ c_1 + c_2 x &= 0. \end{aligned} \tag{1.30}$$

Differentiating 1.30 with respect to x will give us $c_2 = 0$. By substituting $c_2 = 0$ in 1.30, we obtain $c_1 = 0$.

Hence ϕ_1 and ϕ_2 are linearly independent.

Definition 1.7 *Let ϕ_1 , ϕ_2 be two solutions of $L(y) = y'' + a_1 y' + a_2 y = 0$. The determinant*

$$W(\phi_1, \phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}$$
$$= \phi_1\phi_2' - \phi_1'\phi_2$$

is called the Wronskian of ϕ_1, ϕ_2. It is a function, and its value at x is denoted by $W(\phi_1, \phi_2)(x)$.

Theorem 1.6 *Two soutions ϕ_1, ϕ_2 of $L(y) = 0$ are linearly independent on an interval I if and only if $W(\phi_1, \phi_2)(x) \neq 0, \ \forall x \in I$.*

Proof:
Suppose that ϕ_1, ϕ_2 are two solutions of $L(y) = 0$ such that $W(\phi_1, \phi_2)(x) \neq 0, \ \forall x \in I$.
To prove: ϕ_1, ϕ_2 are linearly independent on an interval I.
Let c_1, c_2 be two constants such that

$$c_1\phi_1(x) + c_2\phi_2(x) = 0, \ \forall \ x \in I. \tag{1.31}$$

Then

$$c_1\phi_1'(x) + c_2\phi_2'(x) = 0, \ \forall \ x \in I. \tag{1.32}$$

Now, for a fixed x the equations 1.31 and 1.32 are linearly homogeneous equations satisfied by c_1, c_2. Then the determinant of the coefficients of the equations 1.31 and 1.32 is

$$\begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{vmatrix} = W(\phi_1, \phi_2)(x) \neq 0, \ \forall \ x \in I$$

Thus $c_1 = c_2 = 0$ and hence ϕ_1, ϕ_2 are linearly independent solutions on I.
Conversely suppose that ϕ_1, ϕ_2 are linearly independent on I.
To prove: $W(\phi_1, \phi_2)(x) \neq 0, \ \forall x \in I$.
Suppose that $W(\phi_1, \phi_2)(x_0) = 0$ for some $x_0 \in I$.
Then

$$\begin{vmatrix} \phi_1(x_0) & \phi_2(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) \end{vmatrix} = 0.$$

This implies that the system of two equations

$$c_1\phi_1(x_0) + c_2\phi_2(x_0) = 0 \tag{1.33}$$
$$c_1\phi_1'(x_0) + c_2\phi_2'(x_0) = 0 \tag{1.34}$$

has a solution c_1, c_2, where at least one of these numbers is not zero. Let c_1, c_2 be such a solution and consider the function $\psi = c_1\phi_1 + c_2\phi_2$. Now $L(\psi) = c_1 L(\phi_1) + c_2 L(\phi_2) = 0$ and

$$\psi(x_0) = c_1\phi_1(x_0) + c_2\phi_2(x_0)$$
$$\psi'(x_0) = c_1\phi_1'(x_0) + c_2\phi_2'(x_0).$$

Thus we have, $L(\psi) = 0, \quad \psi(x_0) = 0, \quad \psi'(x_0) = 0$. By uniqueness theorem ψ is the unique solution of the initial value problem, $L(y) = 0, \quad y(x_0) = 0, \quad y'(x_0) = 0$. Therefore $\psi(x) = 0, \ \forall x \in I$ and thus $c_1\phi_1(x) + c_2\phi_2(x) = 0, \forall x \in I$.
Then we have c_1 and c_2 are not both zero such that

$$c_1\phi_1(x) + c_2\phi_2(x) = 0, \ \forall x \in I,$$

which is a contradiction to the fact that ϕ_1 and ϕ_2 are lineraly independent. Therefore $W(\phi_1, \phi_2)(x) \neq 0, \ \forall x \in I$.

Theorem 1.7 *Let ϕ_1, ϕ_2 be two solutions of $L(y) = 0$ on an interval I and let x_0 be any point in I. Then ϕ_1, ϕ_2 are lineraly independent on I if and only if $W(\phi_1, \phi_2)(x) \neq 0$.*

Proof:
Suppose that ϕ_1, ϕ_2 are linearly independent solutions of $L(y) = 0$ on I.
Then by theorem (1.6)
$W(\phi_1, \phi_2)(x) \neq 0, \forall x \in I$.
Since, x_0 in I, then we have $W(\phi_1, \phi_2)(x_0) \neq 0$.
Conversely,
Suppose $W(\phi_1, \phi_2)(x) \neq 0$.
To prove:
ϕ_1, ϕ_2 are linearly independent.
Let c_1 and c_2 be any two constants such that

$$\begin{aligned} c_1\phi_1(x) + c_2\phi_2(x) &= 0 \\ c_1\phi_1'(x) + c_2\phi_2'(x) &= 0 \end{aligned}$$

$\forall x \in I$.
In particular ,

$$\begin{aligned} c_1\phi_1(x_0) + c_2\phi_2(x_0) &= 0 \\ c_1\phi_1'(x_0) + c_2\phi_2'(x_0) &= 0. \end{aligned}$$

Then the determinant of coefficient in the above equation,

$$\begin{vmatrix} \phi_1(x_0) & \phi_2(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) \end{vmatrix} = W(\phi_1, \phi_2)(x_0) \neq 0$$

$\implies c_1 = c_2 = 0$.
Therefore ϕ_1 and ϕ_2 are linearly independent on I.

Theorem 1.8 *Let ϕ_1 , ϕ_2 be any two linearly independent solutions of $L(y) = 0$ on an interval I. Every solution ϕ of $L(y) = 0$ can be written uniquely as $\phi = c_1\phi_1 + c_2\phi_2$, where c_1, c_2 are constants.*

Proof:
Given $\phi = c_1\phi_1 + c_2\phi_2$ be solution of $L(y) = 0$, where c_1, c_2 are constants. Let x_0 be any point in I. Since ϕ_1, ϕ_2 are linearly independent on I, we have $W(\phi_1, \phi_2)(x_0) \neq 0$. Let $\phi(x_0) = \alpha, \phi'(x_0) = \beta$, and consider the two equations,

$$c_1\phi_1(x_0) + c_2\phi_2(x_0) = \alpha$$

$$c_1\phi_1'(x_0) + c_2\phi_2'(x_0) = \beta,$$

where c_1, c_2 are constants. Since the determinant of the coefficients of c_1, c_2 is

$$W(\phi_1, \phi_2)(x_0) \neq 0,$$

there is a unique pair of constants c_1, c_2 satisfying these equations. Choose c_1, c_2 to be these constants. Then the function $\psi = c_1\phi_1 + c_2\phi_2$ is such that

$$\psi(x_0) = \phi(x_0), \ \psi'(x_0) = \phi'(x_0), \ \text{ and } \ L(\psi) = 0.$$

From the uniqueness theorem it follows that $\psi = \phi$ on I, that is, $\phi = c_1\phi_1 + c_2\phi_2$.
Note:

The importance of the previous theorem is that we need only to find any two linearly independent solution on of $L(y) = 0$ in order to obtain all solution of $L(y) = 0$. For example, the equation $y'' + y = 0$ has the two solution e^{ix}, e^{-ix}, which are linearly independent, but it also has the two linearly independent solutions $\cos x, \sin x$.

1.4.1 A formula for the Wronskian

Theorem 1.9 *If ϕ_1, ϕ_2 are two solutions of $L(y) = 0$ on an interval I containing a point x_0, then*

$$W(\phi_1, \phi_2)(x) = e^{-a_1(x-x_0)}W(\phi_1, \phi_2)(x_0).$$

Proof:
Consider $L(y) = y'' + a_1y' + a_2y = 0$.
Since ϕ_1 and ϕ_2 are solutions of $L(y) = 0$, we have $L(\phi_1) = 0$ and $L(\phi_2) = 0$. Then

$$\phi_1'' + a_1\phi_1' + a_2\phi_1 = 0 \tag{1.35}$$
$$\phi_2'' + a_2\phi_2' + a_2\phi_2 = 0. \tag{1.36}$$

Multiply equation 1.35 by $-\phi_2$ will give us

$$\implies -\phi_1''\phi_2 - a_1\phi_1'\phi_2 - a_2\phi_1\phi_2 = 0. \tag{1.37}$$

Multiply equation 1.36 by $-\phi_1$ will give us

$$\implies \phi_2''\phi_1 + a_1\phi_1\phi_2' + a_2\phi_1\phi_2 = 0. \tag{1.38}$$

Adding 1.37 and 1.38, we get

$$(\phi_2''\phi_1 - \phi_1''\phi_2) + a_1(\phi_1\phi_2' - \phi_1'\phi_2) = 0. \tag{1.39}$$

Let $W = W(\phi_1, \phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \phi_1\phi_2' - \phi_1'\phi_2$ and $W'(\phi_1, \phi_2) = \phi_1\phi_2'' - \phi_1''\phi_2$. Then equation 1.39 becomes

$$W' + a_1W = 0,$$

and W satisfies the first order equation. Thus

$$W(x) = ce^{-a_1x},$$

where c is some constant. Let $x = x_0$. Then

$$W(x_0) = ce^{-a_1 x_0},$$

or

$$c = e^{a_1 x_0} W(x_0),$$

and

$$\begin{aligned} W(x) &= e^{a_1 x_0} e^{-a_1 x} W(x_0) \\ W(x) &= e^{-a_1 (x - x_0)} W(x_0). \end{aligned}$$

Hence proved.

Let us sum up

1. We have defined the linearly dependent and independent of a function with some examples.

2. We have discussed the Wronskian definition and formula.

3. We have stated and proved Abel's formula theorem.

4. Finally, we figured out some illustrative examples.

Check your progress

7. Which of the following are linearly independent functions.

   (a) $\phi_1(x) = x$, $\phi_2(x) = e^{rx},\ r$ is a complex constant,

   (b) $\phi_1(x) = \cos x, \phi_2(x) = 3(e^{ix} + e^{-ix})$

   (c) $\phi_1(x) = x^2, \phi_2(x) = 5x^2$

   (d) $\phi_1(x) = \sin x, \phi_2(x) = 4i(e^{ix} - e^{-ix})$

8. The Wronskian of the functions $\phi_1(x) = x^2$, $\phi_2(x) = 5x^2$ is
   (a) -2 (b) -1 (c) 0 (d) 3

9. Define linear independence and dependence.

10. Define Wronskian of two functions ϕ_1 and ϕ_2.

1.5 The non-homogeneous equation of order two

Theorem 1.10 *Let b be continuous on an interval I. Every solution ψ of $L(y) = b(x)$ on I can be written as $\psi = \psi_p + c_1\phi_1 + c_2\phi_2$ where ψ_p is a particular solution. ϕ_1, ϕ_2 are two linearly independent solutions of $L(y) = 0$, and c_1, c_2 are constants. A particular solution ψ_p is given by*

$$\psi_p(x) = \int_{x_0}^{x} \frac{[\phi_1(t)\phi_2(x) - \phi_1(x)\phi_2(t)]b(t)}{W(\phi_1, \phi_2)(t)} dt$$

conversely every such $\psi$is a solution of $L(y) = b(x)$.

Proof: Consider the non-homogeneous equation of order two

$$L(y) = y'' + a_1 y' + a_2 y = b(x),$$

where b is some continuous function on an interval I. Suppose we know that ψ_p is a particular solution of this equation, and that ψ is any other solution. Then,

$$\begin{aligned} \psi &= \psi_p + c_1\phi_1 + c_2\phi_2 \\ \psi - \psi_p &= c_1\phi_1 + c_2\phi_2 \\ L(\psi - \psi_p) &= c_1 L(\phi_1) + c_1 L(\phi_2) \\ L(\psi) - L(\psi_p) &= b - b = 0. \end{aligned}$$

This shows that $\psi - \psi_p$ is a solution of the homogenous equation $L(y) = 0$.
Therefore ϕ_1 and ϕ_1 are linear independent solutions of $L(y) = 0$, there are unique constants such that $\psi - \psi_p = c_1\phi_1 + c_2\phi_2$

$$\psi = \psi_p + c_1\phi_1 + c_2\phi_2.$$

In other words, every solution ψ of $L(y) = b(x)$ can be written in the form

$$\psi = \psi_p + c_1\phi_1 + c_2\phi_2.$$

We see that the problem of finding all solutions of $L(y) = b(x)$ reduces to finding a particular one ψ_p and two linearly independent solution ϕ_1, ϕ_2 of $L(y) = 0$.
If $L(\psi_p) = b$ and $L(\phi_1) = L(\phi_2) = 0$ and c_1, c_2 are any constants, then
$\psi = \psi_p + c_1\phi_1 + c_2\phi_2$ satisfies $L(\phi) = b$.
To find a particular solution of $L(y) = b(x)$ we reason in the following way.
Every solution of $L(y) = 0$ is of the form $c_1\phi_1 + c_2\phi_2$ where c_1, c_2 are constants and ϕ_1, ϕ_2 are linearly independent solutions.
Such a function $c_1\phi_1 + c_2\phi_2$ can not be a solution of $L(y) = b(x)$ unless $b(x) = 0$ on I.
However, suppose we allow c_1, c_2 to u_1, u_2 (not necessarily constants) on I, and then ask whether there is a solution of $L(y) = b(x)$ of the form $u_1\phi_1 + u_2\phi_2$ on I.
This procedure is known as the Variation of constants.
We have a solution of $L(y) = b(x)$ of the form $u_1\phi_1 + u_2\phi_2$ where u_1, u_2 are functions.

$$L(y) = y'' + a_1 y' + a_2 y = b(x).$$

$$\begin{aligned} L(u_1\phi_1 + u_2\phi_2) =& (u_1\phi_1 + u_2\phi_2)'' + a_1(u_1\phi_1 + u_2\phi_2)' + a_2(u_1\phi_1 + u_2\phi) = b(x) \\ =& (u_1'\phi_1 + u_1\phi_1' + u_2'\phi_2 + u_2\phi_2')' + a_1(u_1\phi_1' + u_1'\phi_1 + u_2\phi_2' + u_2'\phi_2) \\ & + a_2(u_1\phi_1 + u_2\phi_2) = b(x) \\ =& u_1'\phi_1' + u_1''\phi_1 + u_1'\phi_1' + u_1\phi_1'' + u_2''\phi_2 + u_2'\phi_2' + u_2\phi_2'' + u_2'\phi_2' \\ & + a_1(u_1\phi_1' + u_1'\phi_1 + u_2\phi_2' + u_2'\phi_2') + a_2(u_1\phi_1 + u_2\phi_2) = b(x) \\ =& u_1[\phi_1'' + a_1\phi_1' + a_2\phi_1] + u_2[\phi_2'' + a_1\phi_2' + a_2\phi_2] + 2[u_1'\phi_1' + u_2'\phi_2'] \\ & + a_1[u_1'\phi_1 + u_2'\phi_2] + u_1''\phi_1 + u_2''\phi_2 = b(x) \\ u_1 L(\phi_1) + u_2 L(\phi_2) + u_1''\phi_1 + u_2''\phi_2 &+ u_2''\phi_2 + 2[u_1'\phi_1' + u_2'\phi_2'] + a_1[u_1'\phi_1 + u_2'\phi_2] = b \end{aligned}$$

Since ϕ_1, ϕ_2 are solutions of $L(y) = 0$
$L(\phi_1) = 0$ and $L(\phi_2) = 0$

$$u_1''\phi_1 + u_2''\phi_2 + 2[u_1'\phi_1' + u_2'\phi_2'] + a_1[u_1'\phi_1 + u_2'\phi_2] = b \tag{1.40}$$

if

$$u_1'\phi_1 + u_2'\phi_2 = 0 \tag{1.41}$$

Differentiate with respect to x

$$u_1''\phi_1 + u_1'\phi_1' + u_2''\phi_2 + u_2'\phi_2' = 0$$
$$u_1''\phi_1 + u_2''\phi_2 = -(u_1'\phi_1' + u_2'\phi_2').$$

Substitute (1.42) and the above equations in (1.41)

$$u_1'\phi_1' + u_2'\phi_2' = b \tag{1.42}$$

$$(6.2)^*\phi_1' \implies u_1'\phi_1\phi_1' + u_2'\phi_2\phi_1' = 0$$
$$(6.4)^*\phi_1 \implies u_1'\phi_1'\phi_1 + u_2'\phi_2'\phi_1 = b\phi_1$$

subtracting the equations we get

$$- u_2'(\phi_2'\phi_1 + \phi_2\phi_1') = -\phi_1 b$$
$$- u_2'.W(\phi_1, \phi_2) = -\phi_1 b$$
$$u_2' = \frac{\phi_1 b}{W(\phi_1, \phi_2)}, \tag{1.43}$$

substitute (1,44) in (1.42)

$$u_1'\phi_1 + u_2'\phi_2 = 0$$
$$u_1'\phi_1 + \frac{\phi_1 b}{W(\phi_1, \phi_2)}\phi_2 = 0$$
$$u_1'\phi_1 = -\frac{\phi_1 b}{W(\phi_1, \phi_2)}\phi_2$$
$$u_1' = -\frac{b}{W(\phi_1, \phi_2)}\phi_2. \tag{1.44}$$

In order to obtain u_1, u_2 all we have to do is to integrate.
Let $x_0 \in$ I and $x > x_0$ Now integrate from x_0 to x,

$$u_2 = \int_{x_0}^{x} \frac{\phi_1(t)b(t)}{W(\phi_1, \phi_2)(t)}dt$$
$$u_1 = -\int_{x_0}^{x} \frac{\phi_2(t)b(t)}{W(\phi_1, \phi_2)(t)}dt.$$

We know that the particular integral is

$$\psi_p = u_1\phi_1(x) + u_1\phi_1(x)$$

$$\psi_p = -\int_{x_0}^{x} \frac{\phi_2(t)b(t)}{W(\phi_1,\phi_2)(t)} dt \phi_1(x) + \int_{x_0}^{x} \frac{\phi_1(t)b(t)}{W(\phi_1,\phi_2)(t)} dt \phi_2(x)$$
$$\psi_p(x) = \int_{x_0}^{x} \frac{\phi_1(t)\phi_2(x) - \phi_2(t)\phi_1(x)}{W(\phi_1,\phi_2)(t)} b(t)dt.$$

Hence proved the theorem.

Example 1.5 *Solve $L(Y) = b(x)$ in the case $p(r) = r^2 + a_1 r + a_2$ has two distinct roots r_1, r_2.*

Solution:
Let $L(y) = y'' + a_1 y' + a_2 y = b(x)$. Given $p(r) = r^2 + a_1 r + a_2$ has two distinct roots r_1, r_2. Therefore $\phi_1(x) = e^{r_1 x}$ and $\phi_2(x) = e^{r_2 x}$ are solutions of $L(y) = 0$. The particular solution ψ_p of $L(y) = b(x)$ is of the form

$$\psi_p = \int_{x_0}^{x} \frac{\phi_1(t)\phi_2(x) - \phi_2(t)\phi_1(x)}{W(\phi_1,\phi_2)(t)} b(t)dt.$$

Here $\phi_1(t) = e^{r_1 t}$ and $\phi_2(t) = e^{r_2 t}$

$$\begin{aligned}
W(\phi_1,\phi_2)(t) &= \begin{vmatrix} \phi_1(t) & \phi_2(t) \\ \phi_1'(t) & \phi_2\prime(t) \end{vmatrix} \\
&= \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} \\
&= r_2 e^{r_2 t} e^{r_1 t} - r_1 e^{r_1 t} e^{r_2 t} \\
&= r_2 e^{(r_1+r_2)t} - r_1 e^{(r_1+r_2)t} \\
&= e^{(r_1+r_2)t}(r_2 - r_1) \\
\psi_p &= \int_{x_0}^{x} \frac{e^{r_1 t} e^{r_2 x} - e^{r_1 x} e^{r_2 t}}{e^{(r_1+r_2)t}(r_2 - r_1)} b(t)dt \\
&= \frac{1}{r_2 - r_1} \int_{x_0}^{x} [e^{r_1 t} e^{r_2 x} - e^{r_1 x} e^{r_2 t}] e^{-(r_1+r_2)t} b(t)dt \\
&= \frac{1}{r_2 - r_1} \int_{x_0}^{x} [e^{r_2 x - r_2 t} - e^{r_1 x - r_1 t}] b(t)dt \\
\psi_p &= \frac{1}{r_2 - r_1} \int_{x_0}^{x} [e^{r_2}(x - t) - e^{r_1}(x - t)] b(t)dt.
\end{aligned}$$

The complete solution is

$$\begin{aligned}
\psi &= \phi_p + c_1\phi_1 + c_2\phi_2 \\
\psi &= \frac{1}{r_2 - r_1} \int_{x_0}^{x} [e_2^r(x - t) - e_1^r(x - t)] b(t)dt + c_1 e^{r_1 x} + c_2 e^{r_2 x}.
\end{aligned}$$

Note:
Suppose we have the solution of $L(y) = b(x)$ of the form $u_1\phi_1 + u_2\phi_2$ where u_1, u_2 are functions. Then u_1', u_2' satisfy the equation,

$$\phi_1 u_1' + \phi_2 u_2' = 0 \text{ and } \phi_1' u_1' + \phi_2' u_2' = b.$$

Let us sum up

1. We have characterized the non-homogeneous equation of order two.

2. We have rectified the particular solution of the non-homogeneous equation of order two using the Wronskian formula.

3. Finally, we figured out some illustrative examples.

Check your progress

11. If ϕ_1 and ϕ_2 are any two solutions of $y'' + a_1y' + a_2y = b(x)$, where a_1, a_2 are constants and $b(x)$ is continuous function on I, then which of the following is a solution of the corresponding homogeneous equation?
(a) $\phi_1 + \phi_2$ (b) $\phi_1 - \phi_2$ (c) $\psi_p(x) + \phi_1 + \phi_2$ (d) None of these

12. The equation is $y'' + yy' = x^2$
(a) linear (b) nonlinear (c) quasi linear (d) semi-linear

Summary

The focus shifts to solving linear ODEs with constant coefficients, with an emphasis on finding general solutions. Topics covered are:

- The concept of the characteristic equation is introduced to find general solutions.

- Distinct real roots, repeated roots, and complex roots.

- Explores second-order linear differential equations, both homogeneous and non-homogeneous.

- An initial values problem for the second order equation consist of finding a solution of the differential equation that also satisfies initial conditions.

- Introduction to Wronskian and its use in determining linear dependence and independence of solutions.

- Solving non-homogeneous equations using the method of variation of parameters.

Glossary

- *Differential equation*: An equation which contains derivatives of one or more depended variables with respect to one or more independent variables.

- *Linear homogeneous second order equation*: A linear homogeneous second order equation with variable coefficients can be written as, $y'' + a_1(x)y' + a_2(x)y = 0$, where $a_1(x)$ and $a_2(x)$ are continuous functions on the interval $[a, b]$.

- *Linearly dependent*: When there is non-zero constants c_1 and c_2 for which the given equation will also be true for all x then we call the two functions linearly dependent.

- *Linearly independent*: When there is only two constants for which the given equation is true for $c_1 = 0$ and $c_2 = 0$ then we call the functions linearly independent.
- *Wronskian*: The Wronskian is a determinant used to check if functions are linearly independent. If the Wronskian is non-zero, the functions are independent; if zero, they might be dependent.

Self-assessment questions

1. The set of linearly independent solutions of the differential equations $\frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} = 0$ is
(a) $\{1, x, e^x, e^{-x}\}$ (b) $\{1, x, e^{-x}, xe^{-x}\}$
(c) $\{1, x, e^x, xe^x\}$ (d) $\{1, x, e^x, xe^{-x}\}$

2. The solution of the differential equation $y'' + y = 0$ satisfying the condition $y(0) = 1, y(\frac{\pi}{2}) = 2$ is
(a) $\phi = \cos x + 2\sin x$ (b) $\phi = \cos x + \sin x$
(c) $\phi = 2\cos x + \sin x$ (d) $\phi = 2\cos x + 2\sin x$

3. Let ϕ_1 and ϕ_2 defined on [0,1] be twice continuously differentiable functions satisfying $y'' + y' + y = 0$. Let $W(x)$ be the Wronskian of ϕ_1 and ϕ_2 and satisfy $W(\frac{1}{2}) = 0$. Then
(a) $W(x) = 0$ for $x \in [0, 1]$ (b) $W(x) \neq 0$ for $x \in [0, 1]$
(c) $W(x) > 0$ for $x \in (\frac{1}{2}, 1]$ (d) None of these.

4. Consider the ordinary differential equation $y'' + P(x)y' + Q(x)y = 0$, where P and Q are smooth functions. Let ϕ_1 and ϕ_2 be any two solutions of the above equation. Let W be the corresponding Wronskian. Which of the following is always true?
(a) If ϕ_1 and ϕ_2 are linearly dependent then there exist x_1,x_2 such that $W(x_1) = 0$ and $W(x_2) \neq 0$.
(b) If ϕ_1 and ϕ_2 are linearly independent then $W(x) = 0\ \forall x$
(c) If ϕ_1 and ϕ_2 are linearly dependent then $W(x) \neq 0\ \forall x$
(d)If ϕ_1 and ϕ_2 are linearly independent then $W(x) \neq 0\ \forall x$

5. Let ϕ_1 and ϕ_2 form a complete set of solutions to the differential equation $y'' - 2xy' + \sin(e^{2x^2})y = 0$, $x \in [0, 1]$ with $\phi_1(0) = 0$, $\phi_1'(0) = 1$, $\phi_2(0) = 1$, $\phi_2'(0) = 1$. The Wronskian $W(x)$ of $\phi_1(x)$ and $\phi_2(x)$ at $x = 0$ is
(a)e^2 (b)$-e^{-1}$ (c)$-e^2$ (d)e

6. Which of the following are linearly independent functions.
(a) $\phi_1(x) = x$, $\phi_2(x) = e^{rx}$, r is a complex constant,
(b) $\phi_1(x) = \cos x, \phi_2(x) = 3(e^{ix} + e^{-ix})$
(c) $\phi_1(x) = x^2, \phi_2(x) = 5x^2$
(d) $\phi_1(x) = \sin x, \phi_2(x) = 4i(e^{ix} - e^{-ix})$

7. Consider the initial value problem in $\mathbb{R}^2$, $y'(t) = Ay + By$; $y(0) = y_0$, where $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$. Then $y(t)$ is given by
(a) $e^{-tA}e^{tB}y_0$ (b) $e^{tB}e^{tA}y_0$ (c) $e^{t(A+B)}y_0$ (d) $e^{-t(A+B)}y_0$

8. Let V be the set of all bounded solution of the ODE $y''(x) - 4y'(x) + 3y(x) = 0$, $x \in \mathbb{R}$. Then V

   (a) is a real vector space of dimension 2.

   (b) is a real vector space of dimension 1.

   (c) contains only the trivial solution.

   (d) contains exactly two solutions.

9. Let ϕ_1 and ϕ_2 defined on [0,1] be twice continuously differentiable functions satisfying $y'' + y' + y = 0$. Let $W(x)$ be the Wronskian of ϕ_1 and ϕ_2 and satisfy $W(\frac{1}{2}) = 0$. Then
   (a) $W(x) = 0$ for $x \in [0, 1]$ (b) $W(x) \neq 0$ for $x \in [0, 1]$
   (c) $W(x) > 0$ for $x \in (\frac{1}{2}, 1]$ (d) None of these.

10. If ϕ_1,ϕ_2 are linearly independent with two solution $L(y) = 0$ on interval I and x_0 be any point on I if and only if,
    (a)$W(\phi_1', \phi_2') \neq 0$ (b)$W(\phi_1, \phi_2) \neq 0$
    (c)$W(\phi_1', \phi_2') = 0$ (d)$W(\phi_1, \phi_2) = 0$

EXERCISES

1. Find all solutions of the following equations:

   (a) $y'' - 4y = 0$

   (b) $3y'' + 2y' = 0$

   (c) $y'' + 16y = 0$

   (d) $y'' = 0$

   (e) $y'' + 2iy' + y = 0$

   (f) $y'' - 4y' + 5y = 0$

   (g) $y'' + (3i - 1)y' - 3iy = 0$.

2. Consider the equation $y'' + y' - 6y = 0$.

   (a) Compute the solution ϕ satisfying $\phi(0) = 1, \phi'(0) = 0$.

   (b) Compute the solution ψ satisfying $\psi(0) = 0, \psi'(0) = 1$.

   (c) Compute $\phi(1)$ and $\psi(1)$.

3. Find all solutions ϕ of $y'' + y = 0$ satisfying:

   (a) $\phi(0) = 1, \phi(\pi/2) = 2$

   (b) $\phi(0) = 0, \phi(\pi) = 0$

   (c) $\phi(0) = 0, \phi'(\pi/2) = 0$

   (d) $\phi(0) = 0, \phi(\pi/2) = 0$.

4. Consider the equation $y'' + k^2y = 0$, where k is a non-negative constant.

(a) For what values of k will there exist non-trivial solutions ϕ satisfying
   (i) $\phi(0) = 0, \phi(\pi) = 0$,
   (ii) $\phi'(0) = 0, \phi'(\pi) = 0$,
   (iii) $\phi(0) = \phi(\pi), \phi'(0) = \phi'(\pi)$,
   (iv) $\phi(0) = -\phi(\pi), \phi'(0) = -\phi'(\pi)$?.

(b) Find the non-trivial solutions for each of the cases (i)-(iv) in (a).

5. Find the solutions of the following initial value problems:
   (a) $y'' - 2y' - 3y = 0,\ y(0) = 0,\ y'(0) = 1,$
   (b) $y'' + (4i + 1)y' + y = 0,\ y(0) = 0,\ y'(0) = 0,$
   (c) $y'' + (3i - 1)y' - 3iy = 0,\ y(0) = 2,\ y'(0) = 0,$
   (d) $y'' + 10y = 0,\ y(0) = \pi,\ y'(0) = \pi^2$.

6. Let I be the interval $0 < x < 1$. Find a function ϕ which has a continuous derivative on $-\infty < x < \infty$, which satisfies

$$y'' = 0 \in I$$
$$y'' + k^2 y = 0 \text{ outside } I, \quad (k > 0),$$

   and which has the form

$$\phi(x) = e^{ikx} + Ae^{-ikx}, \quad (x \leq 0),$$

   and
$$\phi(x) = Be^{ikx}, \quad (x \geq 1).$$

   Determine ϕ by computing the constants A and B, and its values in I.

7. The functions ϕ_1, ϕ_2 defined below exist for $-\infty < x < \infty$. Determine whether they are linearly dependent or independent there.

   (a) $\phi_1(x) = x, \phi_2(x) = e^{rx}$, r is a complex constant

   (b) $\phi_1(x) = \cos x, \phi_2(x) = \sin x$

   (c) $\phi_1(x) = x^2, \phi_2(x) = 5x^2$

   (d) $\phi_1(x) = \sin x, \phi_2(x) = e^{ix}$

   (e) $\phi_1(x) = \cos x, \phi_2(x) = 3(e^{ix} + e^{-ix})$

   (f) $\phi_1(x) = x, \phi_2(x) = |x|$.

8. Are the following statements true or false ? If the statement is true, prove it; if it is false, give a counterexample showing it is false.

   (a) "If ϕ_1, ϕ_2 are linearly independent functions on an interval I, they are linearly independent on any interval J contained inside I."

   (b) "If ϕ_1, ϕ_2 are linearly dependent on an interval I, they are linearly dependent on any interval J contained inside I."

   (c) "If ϕ_1, ϕ_2 are linearly independent solutions of $L(y) = 0$ on an interval I, they are linearly independent on any interval J contained inside I."

(d) "If ϕ_1, ϕ_2 are linearly dependent solutions of $L(y) = 0$ on an interval I, they are linearly dependent on any interval J contained inside I."

9. (a) Show that the functions ϕ_1, ϕ_2 defined by

$$\phi_1(x) = x^2, \quad \phi_2(x) = x|x|,$$

are linearly independent for $-\infty < x < \infty$.

(b) Compute the Wronskian of these functions.

(c) Do the results of parts (a) and (b) contradict Theorem 1.6? Explain your answer.

10. (a) Let ϕ_n be any function satisfying the boundary value problem

$$y'' + n^2 y = 0, \ y(0) = y(2\pi), \ y'(0) = y'(2\pi), \tag{1.45}$$

where $n = 0, 1, 2, \cdots$. Show that

$$\int_0^{2\pi} \phi_n(x)\phi_m(x)dx = 0$$

if $n \neq m$. (Hint: $-\phi_n'' = n^2\phi_n$,and $-\phi_m'' = m^2\phi_m$.Thus

$$(n^2 - m^2)\phi_n\phi_m = \phi_n\phi_m'' - \phi_m\phi_n'' = [\phi_n\phi_m' - \phi_m\phi_n']'.$$

Integrate this equality from 0 to 2π, and use the boundary conditions satisfied by ϕ_n and ϕ_m).

(b) Show that $\cos nx$ and $\sin nx$ are functions satisfying the boundary value problem 1.45. The result of (a) then implies that

$$\int_0^{2\pi} \cos nx \cos mx dx = 0, \quad \int_0^{2\pi} \cos nx \sin mx dx = 0,$$
$$\int_0^{2\pi} \sin nx \sin mx dx = 0, (n \neq m).$$

11. (a) Show that $\phi_n(x) = \sin nx$ satisfies the boundary value problem

$$y'' + n^2 y = 0, \quad y(0) = 0, \quad y(\pi) = 0,$$

where $n = 1, 2, \cdots$.

(b) Using (a) show that

$$\int_0^{\pi} \sin nx \sin mx dx = 0,$$

if $n \neq m$. (Hint: See, Ex. 5 (a)).

(c) Prove that for any positive integer n, $\phi_1, \cdots, \phi_n$ are linearly independent on $0 \leq x \leq \pi$. (Hint: Suppose $a_1\phi_1 + \cdots + a_n\phi_n = 0$. Multiply both sides of equality by ϕ_k (k fixed between 1 and n) and integrate from 0 to 2π. Use (b)).

12. Let ϕ_1, ϕ_2, be two differentiable functions on an interval I, which are not necessarily solutions of an equation $L(y) = 0$. Prove the following:

    (a) If ϕ_1, ϕ_2 are linearly dependent on I, then $W(\phi_1, \phi_2)(x) = 0$ for all x in I.

    (b) If $W(\phi_1, \phi_2)(x) \neq 0$ for some x_0 in I, then ϕ_1, ϕ_2 are linearly independent on I.

    (c) $W(\phi_1, \phi_2)(x) = 0$ for all x in I does not imply that ϕ_1, ϕ_2 are linearly dependent on I.

    (d) $W(\phi_1, \phi_2)(x) = 0$ for all x in I, and $\phi_2(x) \neq 0$ on I, imply that ϕ_1, ϕ_2 are linearly dependent on I. (Hint: Compute $(\frac{\phi_1}{\phi 2})'$).

13. Find all solutions of the following equations:

    (a) $y'' + 4y = \cos x$

    (b) $y'' + 9y = \sin 3x$

    (c) $y'' + y = \tan x, (-\pi/2 < x < \pi/2)$

    (d) $y'' + 2iy' + y = x$

    (e) $y'' - 4y' + 5y = 3e^{-x} + 2x^2$

    (f) $y'' - 7y' + 6y = \sin x$

    (g) $y'' + y = 2 \sin x \sin 2x$

    (h) $y'' + y = \sec x, (-\pi/2 < x < \pi/2)$

    (i) $4y'' - y = e^x$

    (j) $6y'' + 5y' - 6y = x$.

14. Let $L(y) = y'' + a_1 y' + a_2 y$, where a_1, a_2 are constants, and let p be the characteristic polynomial $p(r) = r^2 + a_1 r + a_2$.

    (a) If A, α are constants, and $p(\alpha) \neq 0$, show that there is a solution ϕ of $L(y) = Ae^{ax}$ of the form $\phi(x) = Be^{ax}$, where B is a constant. (Hint: Compute $L(Be^{ax})$).

    (b) Compute a particular solution of $L(y) = Ae^{ax}$ in case $p(\alpha) = 0$ (Hint: If B, r are constants compute $L(Bxe^{rx})$, and then let $r = \alpha$).

    (c) If ϕ, ψ are solutions of

    $$L(y) = b_1(x), \quad L(y) = b_2(x),$$

    respectively, on some interval I, show that $\chi = \phi + \psi$ is a solution of

    $$L(y) = b_1(x) + b_2(x)$$

    on I.

    (d) Suppose $A_1, A_2, \alpha_1, \alpha_2$ are constants, and $p(\alpha_1) \neq 0$, $p(\alpha_2) \neq 0$. Find a solution of

    $$L(y) = A_1 e^{\alpha_1 x} + A_2 e^{\alpha_2 x}.$$

15. Consider
$$L(y) = y'' + a_1 y' + a_2 y,$$
where a_1, a_2 are real constants. Let A, ω be real constants such that $p(i\omega) \neq 0$, where p is the characteristic polynomial.

(a) Show that the equation $L(y) = Ae^{i\omega x}$ has a solution ϕ given by
$$\phi(x) = \frac{A}{|p(i\omega)|} e^{i(\omega x - \alpha)},$$
where $p(i\omega) = |p(i\omega)|e^{i\alpha}$.

(b) If ϕ is any solution of $L(y) = Ae^{i\omega x}$, show that $\phi_1 = Re\ \phi, \phi_2 = Im\ \phi$ are solutions of
$$L(y) = A\cos\omega x, \quad L(y) = A\sin\omega x,$$
respectively.

(c) Using (a), (b) show that there is a particular solution ϕ of
$$Ly'' + Ry' + \frac{1}{C}y = E\cos\omega x,$$
where L, R, C, E, ω are positive constants, which has the form $\phi(x) = B\cos(\omega x - \alpha)$. (Note: L is a constant here, and not a differential operator.)

(d) Suppose that $R^2 C < 2L$ in (c). For what value of ω is B a maximum? (Note: This ω is often referred to as the resonance ω).

16. Consider the equation $y'' + \omega^2 y = A\cos\omega x$, where A, ω are positive constants.

(a) Find all solutions on $0 \leq x \leq \infty$.

(b) Show that every solution ϕ is such that $|\phi(x)|$ assumes arbitrarily large values as $x \to \infty$.

(c) Sketch the graph of that solution ϕ satisfying $\phi(0) = -0, \phi'(0) = 1$.

Answers for check your progress

1. (d) 2. (a) 3. (a) 4. (b)

5. For any real x_0, and constants α, β, there exists a solution ϕ of the initial value problem $L(y) = 0,\ \ y(x_0) = \alpha,\ \ y'(x_0) = \beta.$ on $-\infty < x < \infty$.

6. Let α, β be any two constants, and let x_0 be any real number. On any interval I containing x_0 there exists at most one solution ϕ of the initial value problem
$$L(y) = 0, \ \ y(x_0) = \alpha, \ \ y'(x_0) = \beta.$$

7. (a) 8. (c)

9. Linearly dependent : Two functions ϕ_1, ϕ_2 defined on an interval I are said to be linearly dependent on I, if there exist two constants c_1, c_2, not both zero, such that
$$c_1\phi_1(x) + c_2\phi_2(x) = 0, \ \ \forall\ x \in I.$$

Linearly independent : The functions ϕ_1, ϕ_2 are said to be linearly independent on I, if they are not linearly dependent there. That is, if the only constants c_1, c_2 such that $c_1\phi_1(x) + c_2\phi_2(x) = 0,\ \forall\ x \in I$ are the constants $c_1 = 0, c_2 = 0$.

10. Let ϕ_1 , ϕ_2 be two solutions of $L(y) = y'' + a_1 y' + a_2 y = 0$. The determinant

$$\begin{aligned} W(\phi_1, \phi_2) &= \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} \\ &= \phi_1\phi_2' - \phi_1'\phi_2 \end{aligned}$$

is called the Wronskian of ϕ_1, ϕ_2. It is a function, and its value at x is denoted by $W(\phi_1, \phi_2)(x)$.

11. (a) 12. (b)

Suggested Readings

1. Williams E. Boyce and Richard C. DiPrima, Elementary Differential Equations and Boundary Value Problems, John Wiley and sons, New York, 1967.

2. W. T. Reid, Ordinary Differential Equations, John Wiley and Sons, New York, 1971.

3. Ross, S. L. Differential Equations, 3rd ed. New York: John Wiley and Sons, 1984.

Unit 2

Linear Equations with Constant Coefficients (Continued)

OBJECTIVE:
After going through this unit, you will be able to understand the homogeneous and non-homogeneous equations of order n. Also, we prove the existence and uniqueness results of initial value problems for n^{th} order equations. Further, we discuss the annihilator method for solving the non-homogeneous equation.

2.1 The homogeneous equation of order n

The work we have completed for the second order equation can be applied to the equation of order n as well. Now, let $L(y)$ be defined as

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y,$$

where $a_1, a_2, \cdots, a_n$ are constants. We attempt to solve $L(y) = 0$ using the exponential e^{rx}. We get

$$L(e^{rx}) = p(r)e^{rx}, \tag{2.1}$$

where

$$p(r) = r^n + a_1 r^{n-1} + a_2 r^{n-2} + \cdots + a_n.$$

We refer to p as the characteristic polynomial of L. If r_1 is a root of p, then $L(e^{r_1 x}) = 0$, implying a solution $e^{r_1 x}$. Suppose r_1 is a root of multiplicity m_1 of p. Then

$$p(r_1) = 0, p'(r_1) = 0, \cdots, p^{(m_1-1)}(r_1) = 0. \tag{2.2}$$

If we differentiate the equation 2.1 k times with respect to r, we get

$$\begin{aligned}\frac{\partial^k}{\partial r^k} L(e^{rx}) &= L(\frac{\partial^k}{\partial r^k} e^{rx}) = L(x^k e^{rx}) \\ &= \left[p^{(k)}(r) + kp^{(k-1)}(r)x + \frac{k(k-1)}{2!}p^{(k-2)}(r)x^2 + \ldots + p(r)x^k\right] e^{rx}.\end{aligned}$$

For $k = 0, 1, \cdots, m_1 - 1$, we observe that $x^k e^{r_1 x}$ is a solution of $L(y) = 0$. If k=1 and $r = r_1$, then $L(xe^{r_1 x}) = p'(x_1)e^{r_1 x} + xp(r_1)e^{r_1 x} = 0$.
Therefore $xe^{r_1 x}$ is a solution of $L(y) = 0$. If $k = 2$ and $r = r_1$,

$$\begin{aligned} L(x^2 e^{r_1 x}) &= p'(r_1)e^{r_1 x} + 2xp'(r_1)e^{r_1 x} + x^2 p(r_1)e^{r_1 x} \\ &= 0. \end{aligned}$$

Therefore $x^2 e^{r_1 x}$ is a solution of $L(y) = 0$.

Similarly $L[x^k e^{r_1 x}] = 0$ for $k = 0, 1, 2, \cdots (m_1 - 1)$. Therefore $e^{r_1 x}, xe^{r_1 x}, \cdots, x^{m_1 - 1}e^{r_1 x}$ are solutions of $L(y) = 0$. Repeating this for each root $r_2, r_3, \cdots, r_s$ with multiplicity $m_2, m_3, \cdots, m_s$ we get

$$\begin{gathered} e^{r_2 x}, xe^{r_2 x}, x^2 e^{r_2 x}, \cdots, x^{m_2 - 1}e^{r_2 x}; \\ e^{r_3 x}, xe^{r_3 x}, x^2 e^{r_3 x}, \cdots, x^{m_3 - 1}e^{r_3 x}; \\ \vdots \\ e^{r_s x}, xe^{r_s x}, x^2 e^{r_s x}, \cdots, x^{m_s - 1}e^{r_s x}. \end{gathered}$$

Hence the following result.

Theorem 2.1 *Let $r_1, r_2, \cdots, r_s$ be the distinct roots of the characteristic polynomial p, and suppose r_i has multiplicity m_i (thus $m_1 + m_2 + \ldots + m_s = n$). Then the n functions*

$$\begin{gathered} e^{r_1 x}, xe^{r_1 x}, \cdots, x^{m_1 - 1}e^{r_1 x}; \\ e^{r_2 x}, xe^{r_2 x}, \cdots, x^{m_2 - 1}e^{r_2 x}; \cdots; \\ e^{r_s x}, xe^{r_s x}, \cdots, x^{m_s - 1}e^{r_s x}; \end{gathered}$$

are solutions of $L(y) = 0$.

Definition 2.1 *The n functions $\phi_1, \cdots, \phi_n$ on an interval I are said to be linearly dependent on I if there are constants $c_1, c_2, \cdots, c_n$ not all zero, such that*

$$c_1\phi_1(x) + \cdots + c_n\phi_n(x) = 0,$$

for all x in I. The functions $\phi_1, \phi_2, \cdots, \phi_n$ are said to be linearly independent on I if they are not linearly dependent on I.

Theorem 2.2 *The n solutions of $L(y) = 0$ given in Theorem 2.1 are linearly independent on any interval I.*

Proof:
Suppose we have n constants c_{ij} (for $i = 1, \ldots, s$ and $j = 0, \ldots, m_i - 1$) such that

$$\sum_{i=1}^{s} \sum_{j=0}^{m_i - 1} c_{ij} x^j e^{r_i x} = 0 \tag{2.3}$$

on I. Summing over j for fixed i, we let

$$P_i(x) = \sum_{j=0}^{m_i - 1} c_{ij} x^j \tag{2.4}$$

be the polynomial coefficient of $e^{r_i x}$ in 2.3. Thus we have

$$P_1(x)e^{r_1 x} + P_2(x)e^{r_2 x} + \cdots + P_s(x)e^{r_s x} = 0 \tag{2.5}$$

on I.

Claim: All the c_{ij} are zero

Assume that not all the constants c_{ij} are zero. Then there will be at least one of the polynomials P_i which is not identically zero on I. Assume that $p_s(x)$ is not identically zero on I.

Multiplying equation 2.5 by $e^{-r_1 x}$, we get

$$P_1(x) + P_2(x)e^{(r_2-r_1)x} + \cdots + P_s(x)e^{(r_s-r_1)x} = 0 \tag{2.6}$$

Upon differentiating 2.6 sufficiently many times (at most m_1 times), we can reduce $P_1(x) = 0$. In this process the degree of the polynomials multiplying $e^{(r_s-r_1)x}$ remain unchanged, as well as the non-identically vanishing character of any of these polynomials. We obtain an expression of the form

$$Q_1(x)e^{(r_2-r_1)x} + \cdots + Q_s(x)e^{(r_s-r_1)x} = 0,$$

or

$$Q_1(x)e^{r_2 x} + \cdots + Q_s(x)e^{r_s x} = 0$$

on I, where the Q_i are polynomials, $\deg Q_i = \deg P_i$, and Q_s does not vanish identically. Continuing this process, we finally arrive at a situation where

$$R_s(x)e^{r_s x} = 0 \tag{2.7}$$

on I, and R_s is a polynomial, $\deg R_s = \deg P_s$, which does not vanish identically on I. But 2.7 implies that $R_s(x) = 0$ for all x on I. This contradiction forces us to abandon the supposition that P_s is not identically zero. Thus $P_s(x) = 0$ for all x in I, and we have shown that all the constants $c_{ij} = 0$, proving that the n solutions given in Theorem 2.1 are linearly independent on any interval I.

If $\phi_1, \ldots, \phi_m$ are any m solutions of $L(y) = 0$ on an interval I, and $c_1, \ldots, c_m$ are any m constants, then

$$\phi = c_1\phi_1 + \cdots + c_m\phi_m$$

is also a solution since

$$L(\phi) = c_1 L(\phi_1) + \cdots + c_m L(\phi_m) = 0.$$

As in the case $n = 2$ every solution of $L(y) = 0$ is a linear combination of n linearly independent solutions. The proof of this fact depends on the uniqueness of solutions to initial value problems.

Example 2.1 *Find the solutions of the equation*

$$y''' - 3y' + 2y = 0.$$

Solution:
The characteristic polynomial is $p(r) = r^3 - 3r + 2$. The roots of the characteristic polynomial are $1, 1, -2$. Thus, three linearly independent solutions are given by e^x, xe^x, and e^{-2x}. Any solution $\phi(x)$ of the given differential equation has the form $\phi(x) = (c_1 + c_2x)e^x + c_3e^{-2x}$, where c_1, c_2, and c_3 are any constants.

Let us sum up

1. We have characterized the homogeneous equation of order n.
2. We have defined the linearly dependent and linearly independent homogeneous equations of order n.
3. We have rectified the properties of the homogeneous equation of order n.
4. Finally, we figured out some illustrative examples.

Check your progress

1. The differential equation whose linearly independent solutions are $\cos 2x, \sin 2x$ and e^{-x} is,
(a) $y''' + y'' + 4y' = 0$ (b) $y''' + y'' + 4y' + 4 = 0$
(c) $y''' - y'' + 4y' - 4 = 0$ (d) $y''' - y'' - 4y' + 4 = 0$
2. The general solution for the equation, $y''' - 6y'' + 11y' - 6 = 0$ is
(a) $\phi = c_1e^{-x} + c_2e^{2x} + c_3e^{3x}$ (b) $\phi = c_1e^{x} + c_2e^{-2x} + c_3e^{-3x}$
(c) $\phi = c_1e^{x} + c_2e^{2x} + c_3e^{3x}$ (d) $\phi = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$

2.2 Initial value problems for n-th order equations

An initial value problem for $L(y) = 0$ is a problem of finding a solution ϕ which has prescribed values for it, and its first $n - 1$ derivatives, at some point x_0 (the initial point). If $\alpha_1, \ldots, \alpha_n$ are given constants, and x_0 is some real number, the problem of finding a solution ϕ of $L(y) = 0$ satisfying

$$\phi(x_0) = \alpha_1, \quad \phi'(x_0) = \alpha_2, \cdots, \phi^{(n-1)}(x_0) = \alpha_n,$$

is denoted by

$$L(y) = 0, \quad y(x_0) = \alpha_1, \quad y'(x_0) = \alpha_2, \cdots, y^{(n-1)}(x_0) = \alpha_n.$$

There is only one solution to such an initial value problem, and the demonstration of this will depend on an estimate for the rate of growth of a solution ϕ of $L(y) = 0$, together with its derivatives $\phi', \cdots, \phi^{(n-1)}$. We define

$$\|\phi(x)\| = \left(|\phi(x)|^2 + \cdots + |\phi^{(n-1)}(x)|^2\right)^{1/2},$$

the positive square root being understood.

Theorem 2.3 *Let ϕ be any solution of*

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0$$

on an interval I containing a point x_0. Then for all $x \in I$

$$\|\phi(x_0)\| e^{-k|x-x_0|} \leq \|\phi(x)\| \leq \|\phi(x_0)\| e^{k|x-x_0|}, \tag{2.8}$$

where

$$k = 1 + |a_1| + \cdots + |a_n|.$$

Proof:
Let $u(x) = \|\phi(x)\|^2$, implies

$$u(x) = |\phi(x)|^2 + |\phi'(x)|^2 + \cdots + |\phi^{(n-1)}(x)|^2.$$

Then,

$$\begin{aligned}
u' &= \phi\overline{\phi}' + \phi'\overline{\phi} + \cdots + \phi^{(n-1)}\overline{\phi^{(n)}} + \phi^{(n)}\overline{\phi^{(n-1)}} \\
|u'| &= |\phi\bar{\phi}' + \phi'\bar{\phi} + \phi\bar{\phi}'' + \phi''\bar{\phi}' + \cdots + \phi^{n-1}\bar{\phi}^{n} + \phi^{n}\bar{\phi}^{n-1}| \\
&\leq |\phi\bar{\phi}'| + |\phi'\bar{\phi}| + |\phi\bar{\phi}''| + |\phi''\bar{\phi}'| + \cdots + |\phi^{n-1}\bar{\phi}^{n}| + |\phi^{n}\bar{\phi}^{n-1}| \\
&\leq |\phi||\bar{\phi}'| + |\phi'||\bar{\phi}| + |\phi||\bar{\phi}''| + |\phi''||\bar{\phi}'| + \cdots + |\phi^{n-1}||\bar{\phi}^{n}| + |\phi^{n}||\bar{\phi}^{n-1}| \\
&\leq |\phi||\phi'| + |\phi'||\phi| + |\phi'||\phi''| + |\phi''||\phi'| + \cdots + |\phi^{n-1}||\phi^{n}| + |\phi^{n}||\phi^{n-1}| \\
&\leq 2|\phi||\phi'| + 2|\phi'||\phi''| + \cdots + 2|\phi^{n-1}||\phi^{n}|.
\end{aligned} \tag{2.9}$$

Since ϕ is a solution of $L(y) = 0$, we have$L(\phi) = 0$, that is,

$$\begin{aligned}
\phi^{(n)} + a_1\phi^{(n-1)} + \cdots + a_n\phi = 0 \\
\phi^{n} = -[a_1\phi^{(n-1)} + a_2\phi^{(n-2)} + \cdots + a_n\phi].
\end{aligned} \tag{2.10}$$

Using 2.10 in 2.9, we get

$$\begin{aligned}
|u'| &\leq 2|\phi||\phi'| + 2|\phi'| + |\phi''| + \cdots + 2|\phi^{n-1}|| - [a_1\phi^{(n-1)} + a_2\phi^{(n-2)} + \cdots + a_{(n-1)}\phi' + a_n\phi]| \\
&\leq 2|\phi(x)||\phi'(x)| + 2|\phi'(x)||\phi''(x)| + \cdots + 2|\phi^{n-2}(x)||\phi^{n-1}(x)| + 2|\phi^{n-1}(x)|[|a_1||\phi^{(n-1)}(x)| \\
&+ \cdots + |a_n||\phi(x)|] \\
&\leq 2|\phi(x)||\phi'(x)| + 2|\phi'(x)||\phi''(x)| + \cdots + 2|\phi^{n-2}(x)||\phi^{n-1}(x)| + 2|a_1||\phi^{n-1}(x)|^2 \\
&+ \cdots + 2|a_n||\phi^{(n-1)}(x)||\phi(x)|.
\end{aligned}$$

We now apply the elementary inequality $2|b||c| \leq |b|^2 + |c|^2$ to obtain

$$\begin{aligned}
|u'| &\leq |\phi|^2 + |\phi'|^2 + |\phi'|^2 + |\phi''|^2 + \cdots + |\phi^{n-2}|^2 + |\phi^{n-1}|^2 + |a_2|\left[|\phi^{n-2}|^2 + |\phi^{n-1}|^2\right] + \cdots + |a_n| \\
&\leq (1 + |a_n|)|\phi|^2 + (2 + |a_{n-1}||\phi'|^2) + \cdots + (2 + |a_2|)|\phi^{n-1}|^2 + |\phi^{n-1}|^2 \\
&\leq (2 + 2|a_1| + 2|a_2| + \cdots + 2|a_{n-1}| + 2|a_n|)|\phi|^2 + (2 + 2|a_1| + 2|a_2| + \cdots + 2|a_{n-1}| \\
&+ 2|a_n|)|\phi'|^2 + \cdots + (2 + 2|a_1| + 2|a_2| + \cdots + 2|a_{n-1}| + 2|a_n|)|\phi^{(n-2)}|^2 \\
&\leq (2 + 2|a_1| + 2|a_2| + \cdots + 2|a_{n-1}| + 2|a_n|)[|\phi|^2 + |\phi'|^2 + |\phi''|^2 + \cdots + |\phi^{(n-2)}|^2 + |\phi^{(n-1)}|^2] \\
&\leq 2(1 + |a_1| + |a_2| + \cdots + |a_{n-1}| + |a_n|)u,
\end{aligned}$$

where

$$u = [|\phi|^2 + |\phi'|^2 + |\phi''|^2 + \cdots + |\phi^{(n-2)}|^2 + |\phi^{(n-1)}|^2].$$

Thus

$$|u'(x)| \leq 2ku(x),$$

where $k = (1 + |a_1| + |a_2| + \cdots + |a_{n-1}| + |a_n|)$. Hence

$$-2ku(x) \leq u'(x) \leq 2ku(x). \tag{2.11}$$

Let $x > x_0$. Consider the right inequality of 2.11, we have

$$\begin{aligned} u'(x) &\leq 2ku(x) \\ \frac{u'(x)}{u(x)} &\leq 2k. \end{aligned}$$

Now we integrate from x_0 to x,

$$\begin{aligned} \int_{x_0}^{x} \frac{u'(x)}{u(x)} &\leq 2k \int_{x_0}^{x} dx \\ [\log u(x)]_{x_0}^{x} &\leq 2k[x]_{x_0}^{x} \\ \log u(x) - \log u(x_0) &\leq 2k(x - x_0) \\ \log \left(\frac{u(x)}{u(x_0)} \right) &\leq 2k(x - x_0) \\ \frac{u(x)}{u(x_0)} &\leq 2k(x - x_0) \\ u(x) &\leq e^{2k(x-x_0)} u(x_0). \end{aligned}$$

Substituting $u(x) = \|\phi(x)\|^2$ and taking square root on both sides, we get

$$\|\phi(x)\| \leq \|\phi(x_0)\| e^{k(x-x_0)}. \tag{2.12}$$

Consider the left inequality of 2.11, we have

$$\begin{aligned} -2ku(x) &\leq u'(x) \\ u'(x) &\leq 2ku(x) \\ \frac{u'(x)}{u(x)} &\leq 2k. \end{aligned}$$

Now we integrate from x_0 to x,

$$\begin{aligned} \int_{x_0}^{x} \frac{u'(x)}{u(x)} &\geq -2k \int_{x_0}^{x} dx \\ [\log u(x)]_{x_0}^{x} &\geq -2k[x]_{x_0}^{x} \\ \log u(x) - \log u(x_0) &\geq -2k(x - x_0) \\ \log \left(\frac{u(x)}{u(x_0)} \right) &\leq -2k(x - x_0) \\ \frac{u(x)}{u(x_0)} &\geq -2k(x - x_0) \end{aligned}$$

$$
\begin{aligned}
u(x) &\geq e^{-2k(x-x_0)}u(x_0) \\
\|\phi(x)\| &\geq \|\phi(x_0)\|e^{-k(x-x_0)}.
\end{aligned} \tag{2.13}
$$

From 2.12 and 2.13

$$\|\phi(x_0)\|e^{-k(x-x_0)} \leq \|\phi(x)\| \leq \|\phi(x_0)\|e^{k(x-x_0)}.$$

Similarly for $x < x_0$ by integrating from x to x_0

$$\|\phi(x_0)\|e^{k(x-x_0)} \leq \|\phi(x)\| \leq \|\phi(x_0)\|e^{-k(x-x_0)}$$

from the above two inequalities we can obtain

$$\|\phi(x_0)\|e^{-k|x-x_0|} \leq \|\phi(x)\| \leq \|\phi(x_0)\|e^{k|x-x_0|}.$$

Hence proved the theorem.

Theorem 2.4 *(Uniqueness Theorem) Let $\alpha_1, \cdots, \alpha_n$ be any n constants, and let x_0 be any real number. On any interval I containing x_0, there exists at most one solution ϕ of $L(y) = 0$ satisfying*

$$\phi(x_0) = \alpha_1, \phi'(x_0) = \alpha_2, \cdots, \phi^{(n-1)}(x_0) = \alpha_n.$$

Proof:
We want to prove that there is a unique solution for the initial value problem $L(y) = 0$. Suppose ϕ and ψ were two solutions of the initial value problem $L(y) = 0$ on I satisfying the above conditions at x_0. Let

$$\chi = \phi - \psi. \tag{2.14}$$

Then $L(\chi) = L(\phi) - L(\psi) = 0$ and

$$
\begin{aligned}
\chi(x_0) &= (\phi - \psi)(x_0) \\
&= \alpha_1 - \alpha_1 = 0 \\
\chi'(x_0) &= (\phi - \psi)'(x_0) \\
&= \phi'(x_0) - \psi'(x_0) \\
&= \alpha_2 - \alpha_2 = 0 \\
&\vdots \\
\chi^{(n-1)}(x_0) &= \phi^{(n-1)}(x_0) - \psi^{(n-1)}(x_0) \\
&= \alpha_n - \alpha_n = 0.
\end{aligned}
$$

We know that if ϕ is a solution of $L(y) = 0$, then

$$\|\phi(x_0)\|e^{-k|x-x_0|} \leq \|\phi(x)\| \leq \|\phi(x_0)\|e^{k|x-x_0|},$$

where $k = 1 + |a_1| + \cdots + |a_n|$. Now, applying this inequality for χ we get $\|\chi(x)\| = 0$ for all $x \in I$. This implies $\chi(x) = 0$ for all $x \in I$, or $\phi = \psi$.

Definition 2.2 *The Wronskian $W(\phi_1, \cdots, \phi_n)$ of n functions $\phi_1, \cdots, \phi_n$ having $n-1$ derivatives on an interval I is defined to be the determinant function*

$$W(\phi_1, \cdots, \phi_n) = \begin{vmatrix} \phi_1 & \cdots & \phi_n \\ \phi_1' & \cdots & \phi_n' \\ \vdots & \cdots & \vdots \\ \phi_1^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix},$$

its value at any x in I being $W(\phi_1, \ldots, \phi_n)(x)$.

Theorem 2.5 *If $\phi_1, \ldots, \phi_n$ are n solutions of $L(y) = 0$ on an interval I, they are linearly independent if and only if*

$$W(\phi_1, \ldots, \phi_n)(x) \neq 0, \quad \textit{for all } x \in I.$$

Proof:
Assume that $W(\phi_1, \ldots, \phi_n)(x) \neq 0,$ for all $x \in I$
To prove: $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent.
Let $c_1, c_2, \cdots, c_n$ be constants such that

$$c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x) = 0, \tag{2.15}$$

$\forall x \in I.$

$$c_1\phi_1'(x) + c_2\phi_2'(x) + \cdots + c_n\phi_n'(x) = 0 \tag{2.16}$$

$$\vdots$$

$$c_1\phi_1^{(n-1)}(x) + c_2\phi_2^{(n-1)}(x) + \cdots + c_n\phi_n^{(n-1)}(x) = 0, \tag{2.17}$$

$\forall x \in I$
For fixed x in I, the above n linear homogeneous equations satisfied by $c_1, c_2, \cdots, c_n$. The determinant is $W(\phi_1, \cdots, \phi_n)(x) \neq 0$. Hence there is only one solution to this system namely $c_1 = c_2 = \cdots = c_n = 0$. Therefore $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent.

Conversely assume that $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent solution of $L(y) = 0$.
To prove: $W(\phi_1, \ldots, \phi_n)(x) \neq 0$

Assume that there is an x_0 in I such that $W(\phi_1, \ldots, \phi_n)(x_0) = 0$. This implies that the system of liner equations

$$c_1\phi_1(x_0) + c_2\phi_2(x_0) \quad + \cdots + c_n\phi_n(x_0) = 0 \tag{2.18}$$

$$c_1\phi_1'(x_0) + c_2\phi_2'(x_0) \quad + \cdots + c_n\phi_n'(x_0) = 0 \tag{2.19}$$

$$\vdots$$

$$c_1\phi_1^{(n-1)}(x_0) + c_2\phi_2^{(n-1)}(x_0) \quad + \cdots + c_n\phi_n^{(n-1)}(x_0) = 0 \tag{2.20}$$

has a solution $c_1, c_2, \cdots, c_n$ where not all constants are zero. Let $c_1, c_2, \cdots, c_n$ be such a solution and consider the function $\psi = c_1\phi_1 + c_1\phi_1 + \cdots + c_n\phi_n$. Then

$$L(\psi) = L(c_1\phi_1 + c_1\phi_1 + \cdots + c_n\phi_n)$$

$$
\begin{aligned}
&= c_1L(\phi_1) + c_2L(\phi_2) + \cdots + c_nL(\phi_n) \\
&= 0 \quad \because L(\phi_1) = \cdots L(\phi_n) = 0.
\end{aligned}
$$

Put $x = x_0$. Then

$$
\begin{aligned}
\psi(x_0) &= c_1\phi_1(x_0) + c_2\phi_2(x_0) + \cdots + c_n\phi_n(x_0) = 0 \\
\psi'(x_0) &= c_1\phi_1'(x_0) + c_2\phi_2'(x_0) + \cdots + c_n\phi_n'(x_0) = 0.
\end{aligned}
$$

Similarly

$$\psi''(x_0) = 0 \cdots \psi^{n-1}(x_0) = 0.$$

Thus $\psi(x_0) = 0$. This is contradiction to our assumption that $\phi_1, \ldots, \phi_n$ are linearly independent. Hence $W(\phi_1, \cdots, \phi_n)(x) \neq 0,\ \forall x \in I$.

Theorem 2.6 *(Existence theorem) Let $\alpha_1, \ldots, \alpha_n$ be any n constants, and let x_0 be any real number. There exists a solution ϕ of $L(y) = 0$ on $-\infty < x < \infty$ satisfying*

$$\phi(x_0) = \alpha_1, \phi'(x_0) = \alpha_2, \cdots, \phi^{(n-1)}(x_0) = \alpha_n. \tag{2.21}$$

Proof:
Let $\phi_1, \ldots, \phi_n$ be any set of n linearly independent solutions of $L(y) = 0$ on $-\infty < x < \infty$. It will be shown that there exist unique constants $c_1, \cdots, c_n$ such that

$$\phi = c_1\phi_1 + \cdots + c_n\phi_n$$

is a solution of $L(y) = 0$ satisfying 2.41. Such constants would have to satisfy

$$
\begin{aligned}
c_1\phi_1(x_0) + &\cdots + c_n\phi_n(x_0) = \alpha_1, \\
c_1\phi_1'(x_0) + &\cdots + c_n\phi_n'(x_0) = \alpha_2, \\
&\vdots \\
c_1\phi_1^{(n-1)}(x_0) + &\cdots + c_n\phi_n^{(n-1)}(x_0) = \alpha_n,
\end{aligned}
$$

which is a system of n linear equations for $c_1, \cdots, c_n$. The determinant of the coefficients is just $W(\phi_1, \cdots, \phi_n)(x_0)$ which is not zero, by Theorem 2.5.
Therefore, there is a unique set of constants $c_1, \cdots, c_n$ satisfying the above equations. For this choice of $c_1, \ldots, c_n$, the function

$$\phi = c_1\phi_1 + \cdots + c_n\phi_n$$

will be the desired solution.

Theorem 2.7 *Let $\phi_1, \ldots, \phi_n$ be n linearly independent solutions of $L(y) = 0$ on an interval I. If $c_1, \ldots, c_n$ are any constants,*

$$\phi = c_1\phi_1 + \ldots + c_n\phi_n \tag{2.22}$$

is a solution, and every solution may be represented in this form.

Proof:
We have already seen that

$$L(\phi) = c_1L(\phi_1) + \ldots + c_nL(\phi_n) = 0.$$

Now, let ϕ be any solution of $L(y) = 0$, and let x_0 be in I. Suppose

$$\phi(x_0) = \alpha_1, \phi'(x_0) = \alpha_2, \; \cdots, \phi^{(n-1)}(x_0) = \alpha_n.$$

In the proof of Theorem 2.6, we showed that there exist unique constants $c_1, \cdots, c_n$ such that $\psi = c_1\phi_1 + \cdots + c_n\phi_n$ is a solution of $L(y) = 0$ on I satisfying

$$\psi(x_0) = \alpha_1, \psi'(x_0) = \alpha_2, \cdots, \psi^{(n-1)}(x_0) = \alpha_n.$$

The uniqueness theorem (Theorem 2.4) implies that $\phi = \psi$, proving that every solution ϕ can be represented as in 2.22.

A simple formula exists for the Wronskian, as in the case $n = 2$.

Theorem 2.8 *Let $\phi_1, \ldots, \phi_n$ be n solutions of $L(y) = 0$ on an interval I containing a point x_0. Then*

$$W(\phi_1, \ldots, \phi_n)(x) = e^{-a_1(x-x_0)}W(\phi_1, \ldots, \phi_n)(x_0). \tag{2.23}$$

Proof:
Let $\phi_1, \phi_2, \cdots, \phi_n$ be n solutions of $L(y) = 0$. Then,

$$W(\phi_1, \cdots, \phi_n) = \begin{vmatrix} \phi_1 & \cdots & \phi_n \\ \phi_1' & \cdots & \phi_n' \\ \vdots & \cdots & \vdots \\ \phi_1^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix}.$$

Now W' is sum of n determinants, that is, $W' = v_1 + v_2 + \cdots + v_n$, where v_k differs from W only in its k-th row. v_k is obtained by differentiating k-th row of W.

$$\begin{aligned} W'(X) &= v_1 + v_2 + \cdots + v_n \\ &= \begin{vmatrix} \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{n-1} & \varphi_2^{n-1} & \cdots & \varphi_n^{n-1} \end{vmatrix} + \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1'' & \varphi_2'' & \cdots & \varphi_n'' \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{n-1} & \varphi_2^{n-1} & \cdots & \varphi_n^{n-1} \end{vmatrix} + \cdots + \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{n} & \varphi_2^{n} & \cdots & \varphi_n^{n} \end{vmatrix}. \end{aligned}$$

The 1^{st} $n-1$ determinant $v_1, v_2, \cdots, v_{n-1}$ are all zero because they each have 2 identical rows. Therefore

$$W'(x) = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{n} & \varphi_2^{n} & \cdots & \varphi_n^{n} \end{vmatrix}$$

and

$$\begin{aligned}
L(y) =& y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0 \\
L(\phi_1) =& \phi_1^{(n)} = -a_1\phi_1^{(n-1)} - \cdots - a_n\phi_1 \\
L(\phi_2) =& \phi_2^{(n)} = -a_1\phi_2^{(n-1)} - \cdots - a_n\phi_2 \\
&\vdots \\
L(\phi_n) =& \phi_n^{(n)} = -a_1\phi_n^{(n-1)} - \cdots - a_n\phi_n.
\end{aligned}$$

Thus

$$W' = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \vdots & \vdots & \ddots & \vdots \\ -a_1\phi_n^{(n-1)} - \cdots - a_n\phi_n & -a_2\phi_2^{(n-1)} - \cdots - a_2\phi_n^(n-2) & \cdots & -a_n\phi_1 - \cdots - a_n\phi_n \end{vmatrix}$$

$$= -a_1 \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{n-1} & \varphi_2^{n-1} & \cdots & \varphi_n^{n-1} \end{vmatrix} - a_2 \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{n-2} & \varphi_2^{n-2} & \cdots & \varphi_n^{n-2} \end{vmatrix} + \cdots - a_n \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1 & \varphi_2 & \cdots & \varphi_n \end{vmatrix}.$$

This implies

$$\begin{aligned}
W' &= -a_1 W \\
\frac{W'}{W} &= -a_1 \\
\int \frac{W'}{W} dx &= -\int a_1 dx \\
\log W &= -a_1 x + k, \quad \text{(where } k \text{ is a constant)} \\
W &= e^{-a_1 x + k} \\
&= e^{-a_1 x} e^k \\
&= c e^{-a_1 x}, \quad \text{(where } e^k = c\text{)} \\
W(\phi_1, \phi_2, \cdots, \phi_n)(x) &= c e^{-a_1 x}.
\end{aligned}$$

Put $x = x_0$ in the above equation. Then

$$\frac{W(\phi_1, \phi_2, \cdots, \phi_n)(x_0)}{e^{-a_1 x_0}} = c$$

$$c = W(\phi_1, \phi_2, \cdots, \phi_n)(x_0) e^{a_1 x_0}.$$

Hence

$$\begin{aligned}
W(\phi_1, \phi_2, \cdots, \phi_n)(x) &= W(\phi_1, \phi_2, \cdots, \phi_n)(x_0) e^{a_1 x_0} . e^{-a_1 x} \\
W(\phi_1, \phi_2, \cdots, \phi_n)(x) &= e^{-a_1(x - x_0)} . W(\phi_1, \phi_2, \cdots, \phi_n)(x_0).
\end{aligned}$$

Hence the proof.

Corollary 2.1 *Let $\phi_1, \ldots, \phi_n$ be n solutions of $L(y) = 0$ on an interval I containing x_0. Then they are linearly independent on I if and only if $W(\phi_1, \ldots, \phi_n)(x_0) \neq 0$.*

Let us sum up

1. We have discussed the existence and uniqueness theorem of the initial value problem for n^{th} order equations.

2. We have proved the n solutions of $L(y) = 0$, then the linear combination of those n solutions is also a solution of $L(y) = 0$.

3. Finally, we solved some illustrative examples.

Check your progress

3. Define Wronskian of n functions $\phi_1, \phi_2 \cdots \phi_n$.

4. For the second order differential equations, $||\phi(x)||$ can be defined as
   (a) $|\phi(x)|^2 + |\phi'(x)|^2$ (b) $|\phi'(x)|^2 + |\phi''(x)|^2$
   (c) $[|\phi(x)|^2 + |\phi'(x)|^2]^{\frac{1}{2}}$ (d) $[|\phi'(x)|^2 + |\phi''(x)|^2]^{\frac{1}{2}}$

5. State the existence theorem for solutions of a nth order initial value problem, with constant coefficients.

6. State the uniqueness theorem for solutions of a nth order initial value problem, with constant coefficients.

2.3 The non - homogeneous equation of order n

Theorem 2.9 *Let b be continuous on an interval I, and let $\phi_1, \ldots, \phi_n$ be n-linearly independent solutions of $L(y) = 0$ on I. Every solution ψ of $L(y) = b(x)$ can be written as*

$$\psi = \psi_p + c_1\phi_1 + \cdots + c_n\phi_n,$$

where ψ_p is a particular solution of $L(y) = b(x)$, and $c_1, \ldots, c_n$ are constants. Every such ψ is a solution of $L(y) = b(x)$. A particular solution ψ_p is given by

$$\psi_p(x) = \sum_{k=1}^{n} \phi_k(x) \int_{x_0}^{x} \frac{W_k(t)b(t)}{W(\phi_1, \phi_2, \cdots, \phi_n)(t)} dt. \tag{2.24}$$

Proof:
Let b be a continuous function on an interval I, and consider the equation :

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = b(x), \tag{2.25}$$

where $a_1, a_2, \cdots, a_n$ are constants. If ψ_p is a particular solution of $L(y) = b(x)$, and ψ is any other solution, then

$$L(\psi - \psi_p) = L(\psi) - L(\psi_p) = b(x) - b(x) = 0.$$

Thus $\psi - \psi_p$ is a solution of the homogeneous equation $L(y) = 0$, and this implies that any solution ψ of $L(y) = b(x)$ can be written in the form

$$\psi = \psi_p + c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n,$$

where ψ_p is a particular solution of $L(y) = b(x)$, the functions $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent solutions of $L(y) = 0$, and $c_1, \cdots, c_n$ are constants.

To find a particular solution ψ_p, we proceed just as in the case $n = 2$, that is, we use the variation of constants method. We try to find n functions $u_1, u_2, \ldots, u_n$ so that

$$\psi_p = u_1\phi_1 + u_2\phi_2 + \cdots + u_n\phi_n$$

is a solution. Then

$$\begin{aligned} u_1'\phi_1 + \cdots + u_n'\phi_n &= 0, \\ \psi_p' &= u_1\phi_1' + \cdots + u_n\phi_n', \\ u_1'\phi_1' + \cdots + u_n'\phi_n' &= 0, \\ \psi_p'' &= u_1\phi_1'' + \cdots + u_n\phi_n''. \end{aligned}$$

Thus, if $u_1', \cdots u_n'$ satisfy

$$\begin{aligned} u_1'\phi_1 + \cdots + u_n'\phi_n &= 0, \\ u_1'\phi_1' + \cdots + u_n'\phi_n' &= 0, \\ &\vdots \\ u_1'\phi_1^{(n-2)} + \cdots + u_n'\phi_n^{(n-2)} &= 0, \\ u_1'\phi_1^{(n-1)} + \cdots + u_n'\phi_n^{(n-1)} &= b(x), \end{aligned} \tag{2.26}$$

we see that

$$\begin{aligned} \psi_p &= u_1\phi_1 + \cdots + u_n\phi_n, \\ \psi_p' &= u_1\phi_1' + \cdots + u_n\phi_n', \\ &\vdots \\ \psi_p^{(n-1)} &= u_1\phi_1^{(n-1)} + \cdots + u_n\phi_n^{(n-1)}, \\ \psi_p^{(n)} &= u_1\phi_1^{(n)} + \cdots + u_n\phi_n^{(n)} + b(x). \end{aligned} \tag{2.27}$$

Hence

$$L(\psi_p) = u_1L(\phi_1) + \cdots + u_nL(\phi_n) + b(x) = b(x),$$

and indeed $\psi_p = u_1(\phi_1) + \cdots + u_n(\phi_n)$ is a solution of $L(y) = b(x)$.

Now the determinant of the coefficient

$$W(\phi_1, \ldots, \phi_n)(x) = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \cdots & \varphi_n' \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{n-1} & \varphi_2^{n-1} & \cdots & \varphi_n^{n-1} \end{vmatrix} \neq 0,$$

since $\phi_1, \ldots, \phi_n$ are linearly independent. Therefore there are unique functions $u_1', \cdots, u_n'$ satisfying 2.26. By cramer rule

$$u_1' = \frac{\begin{vmatrix} 0 & \varphi_2 & \cdots & \varphi_n \\ 0 & \varphi_2' & \cdots & \varphi_n' \\ \vdots & \vdots & \ddots & \vdots \\ b & \varphi_2^{n-1} & \cdots & \varphi_n^{n-1} \end{vmatrix}}{W(\phi_1, \ldots, \phi_n)}.$$

In general

$$u_k' = \frac{\begin{vmatrix} \varphi_1 & \varphi_2 & \varphi_{k-1} & 0 & \varphi_{k+1} & \cdots & \varphi_n \\ \varphi_1' & \varphi_2' & \varphi_{k-1}' & 0 & \varphi_{k+1}' & \cdots & \varphi_n' \\ \vdots & \vdots & \ddots & \vdots & & & \\ \varphi_1^{n-1} & \varphi_2^{n-1} & \varphi_{k-1}^{n-1} & b & \varphi_{k+1}^{n-1} & \cdots & \varphi_n^{n-1} \end{vmatrix}}{W(\phi_1, \ldots, \phi_n)}$$

$$u_k'(x) = \frac{W_k(x)b(x)}{W(\phi_1, \ldots, \phi_n)(x)}, \quad k = 1, 2, \cdots, n,$$

where W_k is the determinant obtaind from $W(\phi_1, \ldots, \phi_n)$ by replacing the k-th column by $0, 0, \cdots, 0, 1$. Let x_0 be any point in I. Then

$$u_k(x) = \int_{x_0}^{x} \frac{W_k(t)b(t)}{W(\phi_1, \cdots, \phi_n)(t)} dt, \quad k = 1, 2, \cdots, n.$$

The particular solution is,

$$\psi_p(x) = \sum_{k=1}^{n} \phi_k(x) \int_{x_0}^{x} \frac{W_k(t)b(t)}{W(\phi_1, \phi_2, \cdots, \phi_n)(t)} dt.$$

Example 2.2 *Find the solution of*

$$y''' + y'' + y' + y = 1, \tag{2.28}$$

which satisfies

$$\psi(0) = 0, \ \psi'(0) = 1, \ \psi''(0) = 0. \tag{2.29}$$

Solution:
The homogeneous equation of 2.28 is

$$y''' + y'' + y' + y = 0, \tag{2.30}$$

and the characteristic polynomial corresponding to it is

$$p(r) = r^3 + r^2 + r + 1.$$

The roots of p are $i, -i$, and -1. Since we are interested in a solution satisfying real initial conditions we take for independent solutions of 2.30.

$$\phi_1(x) = \cos x, \ \phi_2(x) = \sin x, \ \phi_3(x) = e^{-x}.$$

To obtain a particular solution of 2.28 of the form $u_1\phi_1 + u_2\phi_2 + u_3\phi_3$ we must solve the following eqations for u_1', u_2', u_3':

$$\begin{aligned}
u_1'\phi_1 + u_2'\phi_2 + u_3'\phi_3 =& 0\\
u_1'\phi_1' + u_2'\phi_2' + u_3'\phi_3' =& 0\\
u_1'\phi_1'' + u_2'\phi_2'' + u_3'\phi_3'' =& 1,
\end{aligned}$$

which in this case reduce to

$$\begin{aligned}
(\cos x)u_1' + (\sin x)u_2' + e^{-x}u_3' =& 0\\
(-\sin x)u_1' + (\cos x)u_2' - e^{-x}u_3' =& 0\\
(-\cos x)u_1' - (\sin x)u_2' + e^{-x}u_3' =& 1.
\end{aligned} \tag{2.31}$$

The determinant of the coefficient is

$$W(\phi_1, \phi_2, \phi_3)(x) = \begin{vmatrix} \cos x & \sin x & e^{-x} \\ -\sin x & \cos x & -e^{-x} \\ -\cos x & -\sin x & e^{-x} \end{vmatrix}.$$

Using 2.23 we have

$$W(\phi_1, \phi_2, \phi_3)(x) = e^{-x}W(\phi_1, \phi_2, \phi_3)(0),$$

since $a_1 = 1$ in this case. Now

$$W(\phi_1, \phi_2, \phi_3)(0) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2,$$

and thus

$$W(\phi_1, \phi_2, \phi_3)(x) = 2e^{-x}.$$

Solving 2.31 for u_1 we find that

$$u_1'(x) = \frac{1}{2}e^{x} \begin{vmatrix} 0 & \sin x & e^{-x} \\ 0 & \cos x & -e^{-x} \\ 1 & -\sin x & e^{-x} \end{vmatrix} = -\frac{1}{2}(\cos x + \sin x). \tag{2.32}$$

Similarly we obtain

$$u_2'(x) = \frac{1}{2}(\cos x - \sin x), \tag{2.33}$$

$$u_3'(x) = \frac{1}{2}e^{x}. \tag{2.34}$$

Integrating 2.32-2.34, we obtain as choice for u_1, u_2, u_3:

$$\begin{aligned}
u_1 =& \frac{1}{2}(\cos x - \sin x),\\
u_2 =& \frac{1}{2}(\sin x + \cos x),
\end{aligned}$$

$$u_3 = \frac{1}{2}e^x.$$

Therefore a particular solution of 2.28 is given by

$$\begin{aligned} u_1(x)\phi_1(x) + u_2(x)\phi_2(x) + u_3(x)\phi_3(x) &= \frac{1}{2}\left[(\cos x - \sin x)\cos x + (\sin x + \cos x)\sin x + 1\right] \\ &= 1. \end{aligned}$$

The general solution of 2.28 is of the form

$$\psi(x) = 1 + c_1 \cos x + c_2 \sin x + c_3 e^{-x},$$

where c_1, c_2, c_3 are constants. We must choose these constants so that the conditions 2.29 are valid. This leads to the g equations for c_1, c_2, c_3:

$$c_1 + c_3 = -1,\ c_2 - c_3 = 1,\ c_1 - c_3 = 0,$$

which have the unique solution

$$c_1 = -\frac{1}{2},\ c_2 = \frac{1}{2},\ c_3 = -\frac{1}{2}.$$

Therefore the solution of the given problem is $\psi(x) = 1 + \frac{1}{2}(\sin x - \cos x - e^{-x})$.

Let us sum up

1. We have characterized and find the particular solution of non-homogeneous equation of order n.

2. Finally, we figured out some illustrative examples.

Check your progress

7. Define nonhomogeneous equations for nth order linear equations with constant coefficients.

2.4 A special method for solving the non-homogeneous equation

Although the variation of constants method yields a solution of the non-homogeneous equation, it sometimes requires more labor than necessary. We now give a method, which is often faster, of solving the non-homogeneous equation $L(y) = b(x)$ when b is a solution of some homogeneous equation $M(y) = 0$ with constant coefficients. Thus $b(x)$ must be a sum of terms of the type $P(x)e^{ax}$, where P is a polynomial and a is a constant. Suppose L and M have constant coefficients, and have orders n and m respectively. If ψ is a solution of $L(y) = b(x)$ and $M(b) = 0$, then clearly

$$M(L(\psi)) = M(b) = 0.$$

This shows that ψ is a solution of a homogeneous equation $M(L(y)) = 0$ with constant coefficients of order $m + n$. Thus ψ can be written as a linear combination with constant coefficients of $m + n$ linearly independent solutions of $M(L(y)) = 0$. Not every linear combination will be a solution of $L(y) = b(x)$, however. Thus, to find out what conditions must be satisfied by the constants, we substitute back into $L(y) = b(x)$. This always leads to a determination of a set of coefficients.

We call this method the annihilator method, since to solve $L(y) = b(x)$, we find an M which annihilates b, i.e., $M(b) = 0$. Once M has been found, the problem becomes algebraic in nature, with no integrations being necessary. Actually, as we have seen from the example, all we require is the characteristic polynomial q of M. The following is a table of some functions together with characteristic polynomials of their annihilators. In this table, a is a constant, and k is a non-negative integer.

Functions	Characteristic Polynomial of an Annihilator
e^{ax}	$r - a$
$x^k e^{ax}$	$(r-a)^{k+1}$
$\sin ax, \cos ax$ (a real)	$r^2 + a^2$
$x^k \sin ax, x^k \cos ax$ (a real)	$(r^2 + a^2)^{k+1}$

Example 2.3 *Find the particular solution of the differential equation*

$$L(y) = y'' - 3y' + 2y = x^2.$$

Solution:
Since x^2 is a solution of $M(y) = y''' = 0$, every solution ψ of $L(y) = x^2$ is a solution of $M(L(y)) = y^{(5)} - 3y^{(4)} + 2y^{(3)} = 0$.

The characteristic polynomial of this equation is $r^3(r^2 - 3r + 2)$, the product of the characteristic polynomials for L and M. The roots are $0, 0, 0, 1, 2$, hence ψ have the form:

$$\psi(x) = c_0 + c_1 x + c_2 x^2 + c_3 e^x + c_4 e^{2x}.$$

We seek a particular solution ψ_p of $L(y) = x^2$, we can assume ψ_p has the form

$$\psi_{\mathfrak{p}}(x) = c_0 + c_1 x + c_2 x^2.$$

To determine c_0, c_1, c_2 such that $L(\psi_p) = x^2$. We yield:

$$\psi_p'(x) = c_1 + 2c_2 x,$$
$$\psi_p''(x) = 2c_2,$$

and

$$L(\psi_p) = (2c_2 - 3c_1 + 2c_0) + (-6c_2 + 2c_1)x + 2c_2 x^2 = x^2.$$

Thus

$$2c_2 = 1, \text{ or } c_2 = 1/2, \text{ and } -6c_2 + 2c_1 = 0, \text{ or } c_2 = 1/2, \text{ and } c_1 = 3/2,$$
$$\text{and } 2c_2 - 3c_1 + 2c_0 = 0, \text{ or } c_0 = 7/4.$$

Therefore, a particular solution is:

$$\psi_p(x) = \frac{1}{4}(7 + 6x + 2x^2)$$

is a particular solution of $L(y) = x^2$.

Let us consider another example of the annihilator method. Consider the equation

$$L(y) = Ae^{(ax)}, \tag{2.35}$$

where L has characteristic polynomial p, and A, a are constants. We assume that a is not a root of p. The operator M given by $M(y) = y' - ay$, with characteristic polynomial $r - a$, annihilates $Ae^{(ax)}$. The characteristic polynomial of ML is $(r-a)p(r)$, and a is a simple root (multiplicity 1) of this. Thus any solution ψ of 2.35 has the form

$$\psi = Be^{(ax)} + \phi,$$

where $L(\phi) = 0$, and B is a constant. Placing back into 2.35 we obtain

$$L(\psi) = BL(e^{(ax)}) + L(\phi) = Bp(a)e^{(ax)} = Ae^{(ax)}.$$

Since $p(a) \neq 0$ we see that $B = A/p(a)$. Therefore we have shown that, if a is not a root of the characteristic polynomial of L, there is a solution ψ of 2.35 of the form

$$\Psi(x) = \frac{A}{p(a)} e^{(ax)}.$$

2.4.1 Algebra of constant coefficient operators

In order to justify the annihilator method, we study the algebra of constant coefficient operators a little more carefully. For the type of equation we have in mind:

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = b(x),$$

where $a_0 \neq 0$, $a_1, \ldots, a_n$ are constants, and b is a sum of products of polynomials and exponentials, every solution ψ has all derivatives on $-\infty < x < \infty$. This follows from the fact that ψ has n derivatives there, and

$$\psi^{(n)} = \frac{b}{a_0} - \frac{a_1}{a_0}\psi^{(n-1)} - \cdots - \frac{a_n}{a_0}\psi,$$

where b has all derivatives on $-\infty < x < \infty$. All the operators we now define will be assumed to be defined on the set of all functions on $-\infty < x < \infty$ which have all derivatives there. Let L and M denote the operators given by:

$$L(\phi) = a_0\phi^{(n)} + a_1\phi^{(n-1)} + \cdots + a_n\phi,$$

$$M(\phi) = b_0\phi^{(m)} + b_1\phi^{(m-1)} + \cdots + b_m\phi,$$

where $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m$ are constants, with $a_0 \neq 0$, $b_0 \neq 0$. It will be convenient in what follows to consider a_0, b_0 which are not necessarily 1. The characteristic polynomials of L and M are thus:

$$p(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_n,$$

$$q(r) = b_0 r^m + b_1 r^{m-1} + \cdots + b_m,$$

respectively. We define the sum $L + M$ to be the operator given by:

$$(L + M)(\phi) = L(\phi) + M(\phi),$$

and the product ML to be the operator given by:

$$(ML)(\phi) = M(L(\phi))$$

If α is a constant, we define αL by:

$$(\alpha L)(\phi) = \alpha(L(\phi)).$$

We note that $L + M$, ML, and αL are all linear differential operators with constant coefficients.

Two operators L and M are said to be equal if

$$L(\phi) = M(\phi)$$

for all ϕ which have an infinite number of derivatives on $-\infty < x < \infty$. Suppose L and M have characteristic polynomials p and q, respectively. Since e^{rx}, for any constant r, has an infinite number of derivatives on $-\infty < x < \infty$, we see that if $L = M$ then

$$L(e^{rx}) = p(r)e^{rx} = M(e^{rx}) = q(r)e^{rx},$$

and hence $p(r) = q(r)$ for all r. This implies that $m = n$ and $a_k = b_k$ for $k = 0, 1, \ldots, n$. Thus $L = M$ if and only if L and M have the same order and the same coefficients, or, equivalently, if and only if $p = q$.

If D is the differentiation operator

$$D(\phi) = \phi',$$

we define $D^2 = DD$ and successively

$$D^k = DD^{k-1}, \quad (k = 2, 3, \cdots).$$

For completeness, we define D^0 by $D^0(\phi) = \phi$, but do not usually write it explicitly. If α is a constant, we understand by α operating on a function ϕ just multiplication by α. Thus

$$\alpha(\phi) = (\alpha D^0)(\phi) = \alpha\phi.$$

Now, using our definitions, it is clear that

$$L = a_0 D^n + a_1 D^{n-1} + \cdots + a_n,$$

and

$$M = b_0 D^m + b_1 D^{m-1} + \cdots + b_m.$$

Theorem 2.10 *The correspondence which associates with each*

$$L = a_0 D^n + a_1 D^{n-1} + \cdots + a_n$$

its characteristic polynomial p given by

$$\rho(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_n$$

is a one-to-one correspondence between all linear differential operators with constant coefficients and all polynomials. If L and M are associated with p and q, respectively, then $L + M$ is associated with $p + q$, ML is associated with pq, and αL is associated with αp (where α is a constant).

Proof:
We have already seen that the correspondence is one-to-one, since $L = M$ if and only if $p = q$. The remainder of the theorem can be shown directly, or by noting that

$$\begin{aligned}(L+M)(e^{rx}) =& L(e^{rx}) + M(e^{rx}) = [p(r)+q(r)]e^{rx},\\ (ML)(e^{rx}) =& M(L(e^{rx})) = M(p(r)e^{rx}) = p(r)M(e^{rx}) = p(r)q(r)e^{rx},\\ (\alpha L)(e^{rx}) =& \alpha(L(e^{rx})) = \alpha p(r)e^{rx}.\end{aligned}$$

This result implies that the algebraic properties of the constant coefficient operators are the same as those of the polynomials. For example, since LM and ML both have the characteristic polynomial pq, we have $LM = ML$. If the roots of p are $r_1, \ldots, r_n$, then

$$p(r) = a_0(r - r_1)\cdots(r - r_n).$$

and since the operator

$$a_0(D - r_1)\cdots(D - r_n)$$

has p as characteristic polynomial, we must have

$$L = a_0(D - r_1)\cdots(D - r_n),$$

since the operator L has p as its characteristic polynomial. This gives a factorization of L into a product of constant coefficient operators of the first order.

Theorem 2.11 *Consider the equation with constant coefficients*

$$L(y) = P(x)e^{(ax)}, \tag{2.36}$$

where P is the polynomial given by

$$P(x) = b_0x^m + b_1x^{(m-1)} + b_m, (b_0 \neq 0). \tag{2.37}$$

Suppose a is a root of the characteristic polynomial p of L of multiplicity j. Then there is a unique solution ψ of 2.36 of the form

$$\psi(x) = x^j(c_0x^m + c_1x^{(m-1)} + cn)e^{ax},$$

where $c_0, c_1, \cdots, c_m$ are constants determined by the annihilator method.

Proof:
The proof makes use of the formula

$$\begin{aligned}L(x^ke^{rz}) = \Bigg[p(r)x^k + kp'(r)x^{k-1} + \frac{k(k-1)}{2!}p''(r)x^{k-2} + \cdots\\ + kp^{(k-1)}(r)x + p^{(k)}(r)\Bigg]e^{rz},\end{aligned} \tag{2.38}$$

which we already proved. The coefficient of $p^{(l)}(r)x^{k-1}$ is the binomial coefficient and it can be written as:

$$\binom{k}{l} = \frac{k!}{(k-l)!l!}.$$

Thus we may write

$$L(x^k e^{rx}) = \left[\sum_{l=0}^{k} \binom{k}{l} p^{(l)}(r) x^{k-l}\right] e^{rx},$$

where we understand $0! = 1$. An annihilator of the right side of 2.36 is

$$M = (D-a)^{m+1},$$

with characteristic polynomial given by

$$q(\tau) = (\tau - a)^{m+1}.$$

Since a is a root of p with multiplicity j, it is a root of pq with multiplicity $j+m+1$. Thus, solutions of $ML(y) = 0$ are of the form

$$\psi(x) = \left(c_0 x^{j+m} + c_1 x^{j+m-1} + \cdots + c_{j+m}\right) e^{ax} + \phi(x),$$

where $L(\phi) = 0$, and ϕ involves exponentials of the form e^{sx} with s a root of p, $s \neq a$. Since a is a root of p with multiplicity j, we have that

$$(c_{m+1} x^{j-1} + c_{m+2} x^{j-2} + \cdots + c_{m+j}) e^{ax}$$

is also a solution of $L(y) = 0$. Consequently, we see that there is a solution of 2.36 having the form

$$\psi(x) = x^j \left(c_0 x^m + c_1 x^{m-1} + \cdots + c_m\right) e^{ax}, \tag{2.39}$$

where $c_0, c_1, \ldots, c_m$ are constants.

We now show that these constants are uniquely determined by the requirement that ψ satisfy 2.36. Substituting 2.39 into L, we obtain

$$L(\psi) = c_0 L(x^{j+m} e^{ax}) + c_1 L(x^{j+m-1} e^{ax}) + \cdots + c_m L(x^j e^{ax}). \tag{2.40}$$

The terms in this sum can be computed using 2.38. We note that

$$p(a) = p'(a) = \cdots = p^{(j-1)}(a) = 0, \; p^{(j)}(a) \neq 0.$$

Since a is a root of p with multiplicity j. We have $k \geq j$:

$$k_j = p(a) x^k + k p'(a) x^{k-1} + \frac{k(k-1)}{2!} p''(a) x^{k-2} + \cdots + p^{(k)}(a),$$

and

$$L(x^{j+n} e^{ax}) = \left[\binom{j+m}{m} p^{(j)}(a) x^m + \binom{j+m}{m-1} p^{(j+1)}(a) x^{m-1} + \cdots + p^{(j+m)}(a)\right] e^{ax}$$

$$L(x^{j+m-1} e^{ax}) = \left[\binom{j+m-1}{m-1} p^{(j)}(a) x^{m-1} + \cdots + p^{(j+m-1)}(a)\right] e^{ax}$$

$$\vdots$$

$$L(x^je^{ax}) = \binom{j}{0} p^j(a)e^{ax} = p^j(a)e^{ax}.$$

Using these computations in 2.40, and noting 2.37, we see that satisfies 2.36 if and only if

$$\begin{aligned} c_0\binom{j+m}{m}p^j(a) &= b_0, \\ c_0\binom{j+m}{m-1}p^{j+1}(a) + c_1\binom{j+m-1}{m-1}p^j(a) &= b_1, \\ &\vdots \\ c_0p^{j+1}(a) + c_1p^{j+m-1}(a) + \cdots + c_mp^j(a) &= b_m. \end{aligned}$$

This is a set of m+1 linear equations for the constants $c_0, c_1, \cdots, c_m$. They have a unique solution, which can be obtained by solving the equations in succession since $p^j(a) \neq 0$. Alternately, we see that the determinant of the coefficients is just

$$\binom{j+m}{m}\binom{j+m-1}{m-1}\cdots 1[p^j(a)]^{m+1} \neq 0.$$

This completes the proof.

Let us sum up

1. We have defined the special method for non-homogeneous equation of order n.
2. We have studied algebra of constant coefficient operators.
3. We have rectified the equation with the constant coefficient by the annihilator method.
4. Finally, we rectified some illustrative examples.

Check your progress

8. The characteristic polynomial of an annihilator method of a function x^ke^{ax} is
   (a) $r-a$ (b) $(r-a)^{k+1}$ (c) $(r^2+a^2)^{k+1}$ (c) r^2+a^2

9. Explain annihilator method.

Summary

This unit provides tools for solving linear differential equations of order n with constant coefficients.

- The homogeneous equations of order 'n' are used for solving the initial value problems for n^{th} order equations.
- Fundamentally for each n^{th} order differential equation the method used involves a set of 'n' linearly independent functions, i.e., a fundamental set of solutions, in order to obtain a general solution.

- The non-homogeneous equations of order n can be solved by finding the particular integral.
- The particular solution to the non-homogeneous equation is evaluated using specific equations and the particular solution.
- The annihilator method is a technique for solving non-homogeneous linear differential equations by applying a differential operator, called the annihilator, that eliminates the non-homogeneous term. Once the equation becomes homogeneous, it is solved, and the particular solution is found using the original non-homogeneous term.

Glossary

- *Initial Value Problem*: It is a differential equation accompanied by an appropriate number of initial conditions and the number of initial conditions essential will depend on the order of the differential equation.
- *Particular solution*: A particular solution is a specific solution to a differential equation that satisfies both the equation and any given initial or boundary conditions. It represents one of possibly many solutions to non-homogeneous differential equations
- *Annihilator method*: The annihilator method is a technique for solving non-homogeneous linear differential equations by applying an operator (annihilator) that turns the non-homogeneous term into zero, allowing you to solve the resulting homogeneous equation.

Self-assessment questions

1. Let W be the Wronskian of two linearly independent solutions of ordinary differential equation $2y'' + y' + t^2y = 0$; $t \in \mathbb{R}$. Then for all t, there exist a constant $C \in \mathbb{R}$ such that $W(t)$ is
   (a) Ce^{-t} (b) $Ce^{\frac{-t}{2}}$ (c) Ce^{2t} (d) Ce^{-2t}
2. Find the **false** statement
   (a) If ϕ_1, ϕ_2 are linearly independent functions on an interval I, they are linearly independent on any interval J contained inside I.
   (b) If ϕ_1, ϕ_2 are linearly dependent on an interval I, they are linearly dependent on any interval J contained inside I.
   (c) If ϕ_1, ϕ_2 are linearly independent solutions of $y'' + c_1y' + c_2y = 0$ on an interval I, then they are linearly independent on any interval J contained inside I.
   (d) If ϕ_1, ϕ_2 are linearly dependent solutions of $y''+c_1y'+c_2y = 0$ on an interval I, then they are linearly dependent on any interval J contained inside I.
3. If ϕ_1 and ϕ_2 are any two solutions of $y'' + a_1y' + a_2y = b(x)$, where a_1, a_2 are constants and $b(x)$ is continuous function on I, then which of the following is a solution of the corresponding homogeneous equation?
   (a) $\phi_1 + \phi_2$ (b) $\phi_1 - \phi_2$ (c) $\psi_p(x) + \phi_1 + \phi_2$ (d) None of these

4. The differential equation whose linearly independent solutions are $\cos 2x, \sin 2x$ and e^{-x} is,
(a) $y''' + y'' + 4y' = 0$ (b) $y''' + y'' + 4y' + 4 = 0$
(c) $y''' - y'' + 4y' - 4 = 0$ (d) $y''' - y'' - 4y' + 4 = 0$

5. Let W be the Wronskian of two linearly independent solutions of ordinary differential equation $2y'' + y' + t^2 y = 0$; $t \in \mathbb{R}$. Then for all t, there exist a constant $C \in \mathbb{R}$ such that $W(t)$ is
(a) Ce^{-t} (b) $Ce^{\frac{-t}{2}}$ (c) Ce^{2t} (d) Ce^{-2t}

6. If ϕ satisfies $y' + 2y = 2 + e^{-x^2}$ with $y(0) = 0$, then $\lim\limits_{x\to\infty}$ equals
(a) 0 (b) 1 (c) 2 (d) -1

7. The Wronskian of the functions $\phi_1(x) = \cos x$, $\phi_2(x) = \sin x$, $\phi_3(x) = e^{-x}$ is
(a) $2e^{-x}$ (b) 2 (c) 3 (d) $2e^{x}$

EXERCISES

1. Are the following sets of functions defined $-\infty < x < \infty$ linearly independent or dependent there? Why?

   (a) $\phi_1(x) = 1, \phi_2(x) = x, \phi_3(x) = x^3$

   (b) $\phi_1(x) = e^{ix}, \phi_2(x) = \sin x, \phi_3(x) = 2\cos x$

   (c) $\phi_1(x) = x, \phi_2(x) = e^{2x}, \phi_3(x) = |x|$.

2. Prove that if p_1, p_2, p_3, p_4 are polynomials of degree two, they are linearly dependent on $-\infty < x < \infty$.

3. Are the following statements true or false? If the statement is true, prove it; otherwise give a counterexample.

   (a) "If $\phi_1, \cdot\cdot\cdot, \phi_n$ are linearly independent functions on an interval I, then any subset of them forms a linearly independent set of functions on I."

   (b) "If $\phi_1, \cdot\cdot\cdot, \phi_n$ are linearly dependent functions on an interval I, then any subset of them forms a linearly dependent set of functions on I."

4. Find all solutions of the following equations:

   (a) $y''' - 8y = 0$

   (b) $y^{(4)} + 16y = 0$

   (c) $y''' - 5y'' + 6y\prime = 0$

   (d) $y''' - iy'' + 4y\prime - 4iy = 0$

   (e) $y^{(100)} + 100y = 0$

   (f) $y^{(4)} + 5y'' + 4y = 0$

   (g) $y^{(4)} - 16y = 0$

   (h) $y''' - 3y\prime - 2y = 0$

(i) $y''' - 3iy'' - 3y\prime + iy = 0$.

5. (a) Compute the Wronskian of four linearly independent solutions of the equation $y^{(4)} + 16y = 0$.

   (b) Compute that the solution ϕ of this equation which satisfies

$$\phi(0) = 1,\ \phi'(0) = 0,\ \phi''(0) = 0,\ \phi'''(0) = 0.$$

6. Find four linearly independent solutions of the equation $y^{(4)} + \lambda y = 0$, in case:

   (a) $\lambda = 0$,

   (b) $\lambda > 0$,

   (c) $\lambda < 0$.

7. Consider the equation

$$y''' - 4y\prime = 0.$$

   (a) Compute three linearly independent solutions.

   (b) Compute the Wronskian of the solutions found in (a).

   (c) Find that solution ϕ satisfying

$$\phi(0) = 0, \quad \phi'(0) = 1, \quad \phi''(0) = 0.$$

8. Consider the equation

$$y^{(5)} - y^{(4)} - y' + y = 0.$$

   (a) Compute five linearly independent solutions.

   (b) Compute the Wronskian of the solutions found in (a), using Theorem 2.8.

   (c) Find that solution ϕ satisfying

$$\phi(0) = 1, \quad \phi'(0) = \phi''(0) = \phi'''(0) = \phi^{(4)}(0) = 0.$$

9. Find all solutions of the following equations:

   (a) $y''' - y' = x$

   (b) $y''' - 8y = e^{ix}$

   (c) $y^{(4)} + 16y = \cos x$

   (d) $y^{(4)} - 4y^{(3)} + 6y'' - 4y' + y = e^x$

   (e) $y^{(4)} - y = \cos x$

   (f) $y'' - 2iy' - y = e^{ix} - 2e^{-ix}$.

10. Consider the equation $L(y) = b(x)$, where b is continuous on an interval I. If $\alpha_1, \cdots, \alpha_n$ are any n constants, and x_0 is a point in I, show that there is exactly one solution ψ of $L(y) = b(x)$ on I satisfying

$$\psi(x_0) = \alpha_1, \psi'(x_0) = \alpha_2, \cdots, \psi^{(n-1)}(x_0) = \alpha_n.$$

(Hint: Let ϕ be the solution of $L(y) = 0$ satisfying the same initial conditions. Let $\psi = \phi + \psi_p$ where ψ_p is given by (2.24). Show that ψ is unique.)

11. Consider the equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = b(x),$$

where $a_1, \cdots, a_n$ are real constants and b is a real-valued continuous function on some interval I. Show that any solution which satisfies real initial conditions is real-valued.

12. Using the annihilator method find a particular solution of each of the following equations:

    (a) $y'' + 4y = \cos x$

    (b) $y'' + 4y = \sin 2x$

    (c) $y'' - 4y = 3e^{2x} + 4e^{-x}$

    (d) $y'' - y' - 2y = x^2 + \cos x$

    (e) $y'' + 9y = x^2 e^{3x}$

    (f) $y'' + y = xe^x \cos 2x$

    (g) $y'' + iy' + 2y = 2\cosh(2x) + e^{-2x}$ $\left(\text{Note} : \cosh u = \frac{(e^u + e^{-u})}{2}\right)$.

    (h) $y''' = x^2 + e^{-x} \sin x$

    (i) $y''' + 3y'' + 3y' + y = x^2 e^{-x}$.

13. Let L be a constant coefficient operator, and suppose ψ_k is a solution of

$$L(y) = b_k(x), \quad k = 1, \cdots, m,$$

where the b_k are continuous functions on some interval I. Show that $\psi = \psi_1 + \psi_2 + \cdots + \psi_m$ is a solution of

$$L(y) = b(x), b = b_1 + \cdots + b_m.$$

14. Suppose $b = b_1 + \cdots + b_m$, where b_k is annihilated by the constant coefficient operator M_k. Show that b is annihilated by $M = M_1 M_2 \cdots M_m$.

15. (a) Show that if f, g are two functions with k derivatives then

$$D^k(fg) = \sum_{l=0}^{k} \binom{k}{l} D^l(f) D^{k-l}(g)$$

where

$$\binom{k}{l} = \frac{k!}{(k-l)!l!}.$$

(b) Show that if g has k derivatives, and r is a constant,

$$D^k(e^{rx} g) = e^{rx}(D + r)^k(g)$$

16. Let L be a linear differential operator with constant coefficients with characteristic polynomial $p(r) = (r-a)^k$, that is $L = (D-a)^k$. Using the result of Ex. 1 (b) show that any solutiion ϕ of $L(y) = 0$ has the form

$$\phi(x) = \epsilon^{(ax)} P(x)$$

where P is a polynomial such that $deg\ P \leq k-1$. Also show that any such ϕ is a solution of $L(y) = 0$.

Answers for check your progress

1. (b) 2. (c)

3. The Wronskian $W(\phi_1, \cdots, \phi_n)$ of n functions $\phi_1, \cdots, \phi_n$ having $n-1$ derivatives on an interval I is defined to be the determinant function

$$W(\phi_1, \cdots, \phi_n) = \begin{vmatrix} \phi_1 & \cdots & \phi_n \\ \phi_1' & \cdots & \phi_n' \\ \vdots & \cdots & \vdots \\ \phi_1^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix},$$

its value at any x in I being $W(\phi_1, \ldots, \phi_n)(x)$.

4. (c)

5. Existence theorem: Let $\alpha_1, \ldots, \alpha_n$ be any n constants, and let x_0 be any real number. There exists a solution ϕ of $L(y) = 0$ on $-\infty < x < \infty$ satisfying

$$\phi(x_0) = \alpha_1, \phi'(x_0) = \alpha_2, \cdots, \phi^{(n-1)}(x_0) = \alpha_n. \quad (2.41)$$

6. Uniqueness Theorem: Let $\alpha_1, \cdots, \alpha_n$ be any n constants, and let x_0 be any real number. On any interval I containing x_0, there exists at most one solution ϕ of $L(y) = 0$ satisfying $\phi(x_0) = \alpha_1, \phi'(x_0) = \alpha_2, \cdots, \phi^{(n-1)}(x_0) = \alpha_n$.

7. Let b be a continuous function on an interval I, and consider the equation :

$$L(y) = y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \cdots + a_n y = b(x),$$

where $a_1, a_2, \cdots, a_n$ are constants.

8. Annihilator method : a technique for solving non-homogeneous linear differential equations by applying a differential operator, called the annihilator, that eliminates the non-homogeneous term. Once the equation becomes homogeneous, it is solved, and the particular solution is found using the original non-homogeneous term.

Suggested Readings

1. E. A. Coddington and N. Levinson. Theory of Ordinary Differential. Equations. New Delhi: Tata Mc Graw-Hill, 1972.

2. W. T. Reid, Ordinary Differential Equations, John Wiley and Sons, New York, 1971.

3. Boyce, W.E. and Richard C. DiPrima. Elementary Differential Equations and Boundary Value Problems. New York: John Wiley and Sons, Inc., 1986.

Unit 3

Linear Equations with Variable Coefficients

OBJECTIVE:
Upon completion of this unit, you will possess the ability to understand the system of linear differential equations and define the method of solution of a known integral and the reduction of the order of a homogeneous equations. Finally, we discuss the significance of Legendre's equations and functions.

3.1 Introduction

A linear differential equation of order n with variable coefficients has the following form:

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x), \tag{3.1}$$

where $a_0, a_1, \cdots, a_n, b$ are complex-valued functions over a real interval I. Points where $a_0(x) = 0$ are known as singular points. Assume that $a_0(x) \neq 0$ on I. Dividing the equation 3.1 by a_0 yields the same equation, but with a_0 substituted by the constant 1. Then we have

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x). \tag{3.2}$$

As in the situation when $a_1, a_2, \cdots, a_n$ are constants, we identify the left side of 3.2 as $L(y)$. Thus

$$L(y) = y^{(n)} + a_1(x)y^{(n-1)} + ... + a_n(x)y, \tag{3.3}$$

and the equation 3.3 becomes $L(y) = b(x)$.

Definition 3.1 *If $b(x) = 0$ for all x on I, then $L(y) = 0$ is called a homogeneous equation, whereas if $b(x) \neq 0$ for some x in I, then $L(y) = b(x)$ is called a non-homogeneous equation.*

We have that L itself is an operator which takes each function ϕ, which has n derivatives on I, into the function $L(\phi)$ on I whose value at x is given by

$$L(\phi)(x) = \phi^{(n)}(x) + a_1(x)\phi^{(n-1)}(x) + \cdots + a_n(x)\phi(x).$$

Thus, a solution of 3.2 on I is a function ϕ on I with n derivatives and $L(\phi) = b$.

In this unit, we assume that the complex-valued functions $a_1, a_2, \cdots, a_n, b$ are continuous on a real interval I, and $L(y)$ denotes the expression 3.3.

3.2 Initial value problems for the homogeneous equation

Although it is not always possible to formulate a solution of 3.1 in terms of simple functions, it can be shown that solutions exist.

Theorem 3.1 *(Existence Theorem) Let $a_1, a_2, \cdots, a_n$ be continuous functions on an interval I containing the point x_0. If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are any n constants, there exists a solution ϕ of*

$$L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0$$

on I satisfying

$$\phi(x_0) = \alpha_1, \phi^{(1)}(x_0) = \alpha_2, \cdots, \phi^{(n-1)}(x_0) = \alpha_n.$$

Theorem 3.2 *Let $b_1, b_2, \cdots, b_n$ be non-negative constants such that for all x in I*

$$|a_j(x)| \leq b_j, \quad (j = 1, 2, \cdots, n),$$

and define k by

$$k = 1 + b_1 + \cdots + b_n.$$

If x_0 is a point in I, and ϕ is a solution of $L(y) = 0$ on I, then

$$||\phi(x_0)||e^{-k|x-x_0|} \leq ||\phi(x)|| \leq ||\phi(x_0)||e^{k|x-x_0|} \tag{3.4}$$

for all x in I.

Proof: Let

$$L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y.$$

Since ϕ is a solution of $L(y) = 0$, we have $L(\phi) = 0$ and

$$\phi^{(n)}(x) + a_1(x)\phi^{(n-1)}(x) + \cdots + a_n(x)\phi(x) = 0.$$

Therefore

$$|\phi^{(n)}(x)| = |-a_1(x)\phi^{(n-1)}(x) - \cdots - a_n(x)\phi(x)|$$
$$|\phi^{(n)}(x)| \leq |a_1(x)\phi^{(n-1)}(x)| + \cdots + |a_n(x)\phi(x)|$$
$$|\phi^{(n)}(x)| \leq |a_1(x)||\phi^{(n-1)}(x)| + \cdots + |a_n(x)||\phi(x)|.$$

Since $|a_j(x)| \leq b_j$, we obtain

$$|\phi^{(n)}(x)| \leq b_1|\phi^{(n-1)}(x)| + \cdots + b_n|\phi(x)|. \tag{3.5}$$

Let

$$u(x) = ||\phi(x)||^2$$

$$
\begin{aligned}
u(x) &= |\phi(x)|^2 + |\phi'(x)|^2 + \cdots + |\phi^{(n-1)}(x)|^2 \\
u(x) &= \phi(x)\overline{\phi(x)} + \phi'(x)\overline{\phi'(x)} + \cdots + \phi^{(n-1)}(x)\overline{\phi^{(n-1)}(x)} \\
u'(x) &= \phi'(x)\overline{\phi(x)} + \phi(x)\overline{\phi'(x)} + \phi''(x)\overline{\phi'(x)} + \phi'(x)\overline{\phi''(x)} + \cdots + \phi^{(n)}(x)\overline{\phi^{(n-1)}(x)} \\
&\quad + \phi^{(n-1)}(x)\overline{\phi^{(n)}(x)} \\
|u'(x)| &\le |\phi'(x)\overline{\phi(x)}| + |\phi(x)\overline{\phi'(x)}| + |\phi''(x)\overline{\phi'(x)}| + |\phi'(x)\overline{\phi''(x)}| + \cdots \\
&\quad + |\phi^{(n)}(x)\overline{\phi^{(n-1)}(x)}| + |\phi^{(n-1)}(x)\overline{\phi^{(n)}(x)}| \\
|u'(x)| &\le |\phi'(x)||\overline{\phi(x)}| + |\phi(x)||\overline{\phi'(x)}| + |\phi''(x)||\overline{\phi'(x)}| + |\phi'(x)||\overline{\phi''(x)}| + \cdots \\
&\quad + |\phi^{(n)}(x)||\overline{\phi^{(n-1)}(x)}| + |\phi^{(n-1)}(x)||\overline{\phi^{(n)}(x)}| \\
|u'(x)| &\le 2|\phi(x)||\phi'(x)| + \cdots + 2|\phi^{(n-1)}(x)|[b_1|\phi^{(n-1)}(x)| + b_2|\phi^{(n-2)}(x)| + \cdots + b_n|\phi(x)|] \\
&\le 2|\phi(x)||\phi'(x)| + 2|\phi'(x)||\phi''(x)| + \cdots + 2b_1|\phi^{(n-1)}(x)||\phi^{(n-1)}(x)| \\
&\quad + 2b_2|\phi^{(n-1)}(x)||\phi^{(n-2)}(x)| + \cdots + 2b_n|\phi^{(n-1)}(x)||\phi(x)|.
\end{aligned}
$$

Using the result $2|b||c| \le |b|^2 + |c|^2$, we obtain

$$
\begin{aligned}
|u'(x)| &\le |\phi(x)|^2 + |\phi'(x)|^2 + |\phi'(x)|^2 + |\phi''(x)|^2 + \cdots + b_1|\phi^{(n-1)}(x)|^2 + b_1|\phi^{(n-1)}(x)|^2 \\
&\quad + b_2|\phi^{(n-1)}(x)|^2 + b_2|\phi^{(n-2)}(x)|^2 + \cdots + b_n|\phi^{(n-1)}(x)|^2 + b_n|\phi(x)|^2 \\
&\le (1+b_n)|\phi(x)|^2 + (2+b_{n-1})|\phi'(x)|^2 + \cdots + (1+2b_1+b_2+\cdots+b_n)|\phi^{n-1}(x)|^2 \\
&\le (2+2b_1+2b_2+\cdots+2b_n)|\phi(x)|^2 + (2+2b_1+2b_2+\cdots+2b_n)|\phi'(x)|^2 + \cdots \\
&\quad + (2+2b_1+2b_2+\cdots+2b_n)|\phi^{(n-1)}(x)|^2 \\
&\le 2(1+b_1+b_2+\cdots+b_n)[|\phi(x)|^2 + |\phi'(x)|^2 + \cdots + |\phi^{(n-1)}(x)|^2] \\
|u'(x)| &\le 2(1+b_1+b_2+\cdots+b_n)u(x) \\
|u'(x)| &\le 2ku(x) \\
-2ku(x) &\le u'(x) \le 2ku(x).
\end{aligned}
$$

Let $x > x_0$. Consider the right side of the above inequality,

$$
\begin{aligned}
u'(x) &\le 2ku(x) \\
\frac{u'(x)}{u(x)} &\le 2k \\
\int_{x_0}^{x} \frac{u'(x)}{u(x)}dx &\le 2k\int_{x_0}^{x} dx \\
[\log u(x)]_{x_0}^{x} &\le 2k[x]_{x_0}^{x} \\
\log u(x) - \log u(x_0) &\le 2k(x-x_0) \\
\log\frac{u(x)}{u(x_0)} &\le 2k(x-x_0) \\
\frac{u(x)}{u(x_0)} &\le e^{2k(x-x_0)} \\
u(x) &\le e^{2k(x-x_0)}u(x_0) \\
||\phi(x)||^2 &\le e^{2k(x-x_0)}||\phi(x_0)||^2
\end{aligned}
$$

Taking square root on both sides,

$$||\phi(x)|| \leq e^{k(x-x_0)}||\phi(x_0)||. \tag{3.6}$$

Similarly, if we consider the left side of the inequality, then we obtain

$$||\phi(x)|| \geq e^{-k(x-x_0)}||\phi(x_0)||. \tag{3.7}$$

From 3.6 and 3.7, we have

$$e^{-k(x-x_0)}||\phi(x_0)|| \leq ||\phi(x)|| \leq e^{k(x-x_0)}||\phi(x_0)||.$$

Similarly, we can prove the inequality for $x < x_0$. Finally, we get

$$||\phi(x_0)||e^{-k|x-x_0|} \leq ||\phi(x)|| \leq ||\phi(x_0)||e^{k|x-x_0|}.$$

Theorem 3.3 *(Uniqueness Theorem) Let x_0 be in I, and let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be any n constants. There is at most one solution ϕ of $L(y) = 0$ on I satisfying*

$$\phi(x_0) = \alpha_1, \phi'(x_0) = \alpha_2, \cdots, \phi^{(n-1)}(x_0) = \alpha_n. \tag{3.8}$$

Proof:
Let ϕ and ψ be two solutions of $L(y) = 0$ on I satisfying the condition 3.8 at x_0.

$$\psi(x_0) = \alpha_1, \psi'(x_0) = \alpha_2, \cdots, \psi^{(n-1)}(x_0) = \alpha_n. \tag{3.9}$$

Consider $\chi = \phi - \psi$.
To prove: $\chi(x) = 0$ for all x on I.
Even though the functions a_j are continuous on I they need not be bounded there. However let $x \neq x_0$ be any point on I and let J be any closed bounded interval in I which contains x_0 and x. On this interval the functions a_j are bounded, that is,

$$|a_j(x)| < b_j, \qquad (j = 1, 2, \cdots, n),$$

on J for some constant b_j, which may depend on J. Now we can apply above theorem to χ defined on J. We have $L(\chi) = 0$ on J, and

$$\begin{aligned}
\chi(x) &= \phi(x) - \psi(x) \\
\chi(x_0) &= \phi(x_0) - \psi(x_0) \\
&= \alpha_1 - \alpha_1 \\
&= 0. \\
\chi'(x_0) &= \phi'(x_0) - \psi'(x_0) \\
&= \alpha_2 - \alpha_2 \\
&= 0. \\
&\vdots \\
\chi^{(n-1)}(x_0) &= \phi^{(n-1)}(x_0) - \psi^{(n-1)}(x_0) \\
&= \alpha_n - \alpha_n.
\end{aligned}$$

This implies, $||\chi(x_0)|| = 0$.
By previous theorem,

$$\begin{aligned} ||\chi(x_0)||e^{-k|x-x_0|} \leq ||\chi(x)|| &\leq ||\chi(x_0)||e^{k|x-x_0|} \\ 0 \leq ||\chi(x)|| &\leq 0 \\ ||\chi(x)|| &= 0 \\ \implies \chi(x) &= 0, \ \forall x \in I \\ \implies \phi(x) - \psi(x) &= 0, \ \forall x \in I. \end{aligned}$$

Hence

$$\phi(x) = \psi(x), \ \forall x \in I.$$

3.2.1 Solutions of the homogeneous equation

If $\phi_1, \phi_2, \cdots, \phi_m$ are any m solutions of n-th order equation $L(y) = 0$ on an interval I, and $c_1, c_2, \cdots, c_m$ are any m constants, then

$$L(c_1\phi_1 + c_2\phi_2 + \cdots + c_m\phi_m) = c_1L(\phi_1) + c_2L(\phi_2) + \cdots + c_mL(\phi_m),$$

which implies that $c_1\phi_1 + c_2\phi_2 + \cdots + c_m\phi_m$ is also a solution. That is, any linear combination of solutions is again a solution. The trivial solution is the function which is identically zero on I.

As in the case of an L with constant coefficients, every solution of $L(y) = 0$ is a linear combination of any n linearly independent solutions.

Definition 3.2 *The n functions $\phi_1, \phi_2, \cdots, \phi_n$ defined on an interval I are said to be linearly independent if the only constants $c_1, c_2, \cdots, c_n$, such that*

$$c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x) = 0,$$

for all x in I are the constants $c_1 = c_2 = \cdots = c_n = 0$.

Theorem 3.4 *There exist n linearly independent solutions of $L(y) = 0$ on I.*

Proof:
Let x_0 be a point in I. According to Theorem 3.1 there is a solution ϕ_1 of $L(y) = 0$ satisfying

$$\phi_1(x_0) = 1, \phi_1'(x_0) = 0, \cdots, \phi_1^{(n-1)}(x_0) = 0.$$

In general for each $i = 1, 2, \cdots, n$ there is a solution ϕ_i satisfying

$$\phi_i^{(i-1)}(x_0) = 1, \phi_i^{(j-1)}(x_0) = 0, \ j \neq i. \tag{3.10}$$

The solutions $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent on I, for suppose there are constants $c_1, c_2, \cdots, c_n$ such that

$$c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x) = 0, \tag{3.11}$$

for all x in I. Differentiating we get

$$\begin{aligned} c_1\phi_1'(x) + c_2\phi_2'(x) + \cdots + c_n\phi_n'(x) =& 0 \\ c_1\phi_1''(x) + c_2\phi_2''(x) + \cdots + c_n\phi_n''(x) =& 0 \\ & \vdots \\ c_1\phi_1^{(n-1)}(x) + c_2\phi_2^{(n-1)}(x) + \cdots + c_n\phi_n^{(n-1)}(x) =& 0 \end{aligned} \tag{3.12}$$

for all x in I. In particular, the equations 3.11, 3.12 must hold at x_0. Putting $x = x_0$ in 3.11 we find, using 3.10 that $c_1 1 + 0 + \cdots + 0 = 0$, or $c_1 = 0$. Putting $x = x_0$ in the equations 3.12 we obtain $c_2 = c_3 = \cdots = c_n = 0$, and thus the solutions $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent.

Theorem 3.5 *Let $\phi_1, \phi_2, \cdots, \phi_n$ be the n solutions of $L(y) = 0$ on I satisfying 3.10. If ϕ is any solution of $L(y) = 0$ on I, then there are n constants $c_1, c_2, \cdots, c_n$ such that*

$$\phi = c_1\phi_1 + \cdots + c_n\phi_n.$$

Proof:
Let

$$\phi(x_0) = \alpha_1, \phi'(x_0) = \alpha_2, \cdots, \phi^{(n-1)}(x_0) = \alpha_n,$$

and consider the function

$$\psi = \alpha_1\phi_1 + \alpha_2\phi_2 + \cdots + \alpha_n\phi_n.$$

It is solution of $L(y) = 0$, and clearly

$$\psi(x_0) = \alpha_1\phi_1(x_0) + \alpha_2\phi_2(x_0) + \cdots + \alpha_n\phi_n(x_0) = \alpha_1,$$

since

$$\phi_1(x_0) = 1, \phi_2(x_0) = 0, \cdots, \phi_n(x_0) = 0.$$

Using the other relation in 3.10 we see that

$$\psi(x_0) = \alpha_1, \psi'(x_0) = \alpha_2, \cdots, \psi^{(n-1)}(x_0) = \alpha_n.$$

Thus ψ is a solution of $L(y) = 0$ having the same initial conditions at x_0 as ϕ. By uniqueness theorem, we must have $\phi = \psi$, that is, $c_1 = \alpha_1, c_2 = \alpha_2, \cdots, c_n = \alpha_n$.

Definition 3.3 *A set of functions which has the property that, if ϕ_1, ϕ_2 belong to the set, and c_1, c_2 are any two constants, then $c_1\phi_1 + c_2\phi_2$ belongs to the set also, is called a Linear space of functions.*

We have just seen that the set of all solutions of $L(y) = 0$ on an interval I is a linear space of functions.

Definition 3.4 *If a linear space of functions contains n functions $\phi_1, \cdots, \phi_n$ which are linearly independent and such that every function in the space can be represented as a linear combination of these, then $\phi_1, \cdots, \phi_n$ is called a basis for the linear space, and the dimension of the linear space is the integer n.*

The above theorem states that the functions $\phi_1, \cdots, \phi_n$ satisfying the initial conditions 3.10 form a basis for the solutions of $L(y) = 0$ on I, and this linear space of functions has dimension n.

Let us sum up

1. We have defined linear differential equation of order n with variable coefficients.
2. We have discussed the existence and uniqueness theorem of the initial value problem for linear differential equation of order n with variable coefficients.
3. We have proved the inequality for $x < x_0$, we get

$$||\phi(x_0)||e^{-k|x-x_0|} \leq ||\phi(x)|| \leq ||\phi(x_0)||e^{k|x-x_0|}.$$

4. We have characterized the any linear combination of solutions is again a solution.
5. Finally, we defined the linear space and basis of the linear space.

Check your progress

1. State the existence theorem for solutions of a nth order initial value problem, with variable coefficients.
2. State the uniqueness theorem for solutions of a nth order initial value problem, with variable coefficients.

3.3 The Wronskian and linear independence

To demonstrate that any set of n linearly independent solutions of $L(y) = 0$ can serve as a basis for the solutions of $L(y) = 0$, we consider the Wronskian $W(\phi_1, \phi_2, \cdots, \phi_n)$. Remember that this is defined as the determinant

$$W(\phi_1, \phi_2, \cdots, \phi_n) = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & \cdots & \vdots \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix}.$$

Theorem 3.6 *If $\phi_1, \phi_2, \cdots, \phi_n$ are n solutions of $L(y) = 0$ on an interval I, then they are linearly independent there if, and only if,*

$$W(\phi_1, \phi_2, \cdots, \phi_n) \neq 0, \ \forall x \in I.$$

Proof:
First suppose $W(\phi_1, \phi_2, \cdots, \phi_n)(x) \neq 0$ for all x in I. If there are constants $c_1, c_2, \cdots, c_n$ such that

$$c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x) = 0 \tag{3.13}$$

for all x in I, then clearly

$$\begin{aligned} c_1\phi_1'(x) + c_2\phi_2'(x) + \cdots + c_n\phi_n'(x) &= 0 \\ c_1\phi_1''(x) + c_2\phi_2''(x) + \cdots + c_n\phi_n''(x) &= 0 \\ &\vdots \\ c_1\phi_1^{(n-1)}(x) + c_2\phi_2^{(n-1)}(x) + \cdots + c_n\phi_n^{(n-1)}(x) &= 0 \end{aligned} \tag{3.14}$$

for all x in I. For a fixed x in I the equations 3.13, 3.14 are n linear homogeneous equations satisfied by $c_1, c_2, \cdots, c_n$. The determinant of the coefficients is just $W(\phi_1, \phi_2, \cdots, \phi_n)(x)$, which is not zero. Hence there is only one solution to this system, namely $c_1 = c_2 = \cdots = c_n = 0$.
Therefore $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent on I.

Conversely, suppose $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent on I. Suppose there is an x_0 in I such that

$$W(\phi_1, \phi_2, \cdots, \phi_n)(x_0) = 0.$$

Then this implies that the system if n linear equations

$$\begin{aligned} c_1\phi_1(x_0) + c_2\phi_2(x_0) + \cdots + c_n\phi_n(x_0) &= 0 \\ c_1\phi_1'(x_0) + c_2\phi_2'(x_0) + \cdots + c_n\phi_n'(x_0) &= 0 \\ &\vdots \\ c_1\phi_1^{(n-1)}(x_0) + c_2\phi_2^{(n-1)}(x_0) + \cdots + c_n\phi_n^{(n-1)}(x_0) &= 0 \end{aligned} \tag{3.15}$$

has a solution $c_1, c_2, \cdots, c_n$ where not all the constants $c_1, c_2, \cdots, c_n$ are zero. Let $c_1, c_2, \cdots, c_n$ be such a solution, and consider the function

$$\psi = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n.$$

Now $L(\psi) = 0$, and from 3.15 we get

$$\psi(x_0) = 0, \psi'(x_0) = 0, \cdots, \psi^{(n-1)}(x_0) = 0.$$

From the uniqueness theorem it follows that $\psi(x) = 0$ for all $x \in I$, and thus

$$c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x) = 0,$$

for all x in I. But this contradicts the fact that $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent on I. Thus the supposition that there was a point x_0 in I such that

$$W(\phi_1, \phi_2, \cdots, \phi_n)(x_0) = 0$$

must be false. We have consequently proved that

$$W(\phi_1, \phi_2, \cdots, \phi_n)(x) \neq 0, \ \forall x \in I.$$

Theorem 3.7 *Let $\phi_1, \phi_2, \cdots, \phi_n$ be n linearly independent solutions of $L(y) = 0$ on an interval I. If ϕ is any solution of $L(y) = 0$ on I, it can be represented in the form*

$$\phi = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n$$

where $c_1, c_2, \cdots, c_n$ are constants. Thus any set of n linearly independent solutions of $L(y) = 0$ on I is a basis for the solutions of $L(y) = 0$ on I.

Proof:
Let x_0 be a point in I, and suppose

$$\phi(x_0) = \alpha_1, \phi'(x_0) = \alpha_2, \cdots, \phi^{(n-1)}(x_0) = \alpha_n.$$

We show that there exist unique constants $c_1, c_2, \cdots, c_n$ such that

$$\psi = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n$$

is a solution of $L(y) = 0$ satisfying

$$\psi(x_0) = \alpha_1, \psi'(x_0) = \alpha_2, \cdots, \psi^{(n-1)}(x_0) = \alpha_n.$$

By the uniqueness theorem we then have $\phi = \psi$, or

$$\phi = c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n.$$

The initial conditions for ψ are equivalent to the following equations for $c_1, c_2, \cdots, c_n$:

$$\begin{aligned} c_1\phi_1(x_0) + c_2\phi_2(x_0) + \cdots + c_n\phi_n(x_0) =& \alpha_1 \\ c_1\phi_1'(x_0) + c_2\phi_2'(x_0) + \cdots + c_n\phi_n'(x_0) =& \alpha_2 \\ &\vdots \\ c_1\phi_1^{(n-1)}(x_0) + c_2\phi_2^{(n-1)}(x_0) + \cdots + c_n\phi_n^{(n-1)}(x_0) =& \alpha_n. \end{aligned} \tag{3.16}$$

This is set of n linear equations for $c_1, c_2, \cdots, c_n$. The determinant of the coefficients is $W(\phi_1, \phi_2, \cdots, \phi_n)(x_0)$, which is not zero since $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent (Theorem 3.6). Therefore there is a unique solution $c_1, c_2, \cdots, c_n$ of the equation 3.16, and this completes the proof.

Theorem 3.8 *Let $\phi_1, \phi_2, \cdots, \phi_n$ be n solutions of $L(y) = 0$ on an interval I, and let x_0 be any point in I. Then*

$$W(\phi_1, \phi_2, \cdots, \phi_n)(x) = \exp\left[-\int_{x_0}^{x} a_1(t)dt\right] W(\phi_1, \phi_2, \cdots, \phi_n)(x_0). \tag{3.17}$$

Proof:
We first prove this result for the simple case $n = 2$, and then give a proof which is valid for general n.The latter proof makes use of some general properties of determinants.
Proof for the case $n = 2$:
In this case

$$W(\phi_1, \phi_2) = \phi_1\phi_2' - \phi_2\phi_1',$$

and therefore

$$\begin{aligned} W'(\phi_1, \phi_2) &= \phi_1'\phi_2' + \phi_1\phi_2'' - \phi_1''\phi_2 - \phi_1'\phi_2' \\ &= \phi_1\phi_2'' - \phi_1''\phi_2. \end{aligned}$$

Since ϕ_1, ϕ_2 satisfy $y'' + a_1(x)y' + a_2(x)y = 0$, we get

$$\begin{aligned} \phi_1'' &= -a_1\phi_1' - a_2\phi_1, \\ \phi_2'' &= -a_1\phi_2' - a_2\phi_2. \end{aligned}$$

Then

$$\begin{aligned} W'(\phi_1, \phi_2) =&\phi_1(-a_1\phi_2' - a_2\phi_2) - (-a_1\phi_2' - a_2\phi_1)\phi_2 \\ =& - \phi_1 a_1\phi_2' - \phi_1 a_2\phi_2 + \phi_2 a_1\phi_2' + \phi_2 a_2\phi_1 \\ =& - a_1(\phi_1\phi_2' - \phi_1'\phi_2) \\ =& - a_1 W(\phi_1, \phi_2) \\ W'(\phi_1, \phi_2) + a_1 W(\phi_1, \phi_2) =& 0. \end{aligned}$$

We see that $W(\phi_1, \phi_2)$ satisfies the linear first order equation $y' + a_1(x)y = 0$, and hence

$$W(\phi_1, \phi_2)(x) = c\exp\left[-\int_{x_0}^{x} a_1(t)dt\right],$$

where c is a constant. By putting $x = x_0$, we obtain

$$c = W(\phi_1, \phi_2)(x_0),$$

thus proves 3.17 for the case $n = 2$.

Proof for a general n:
We let $W = W(\phi_1, \phi_2, \cdots, \phi_n)$ for brevity. From the definition of W as a determinant it follows that its derivative W' is a sum of n determinants

$$W' = V_1 + V_2 + \cdots + V_n,$$

where V_k differs from W only in its k-th row, and the k-th tow of V_k is obtained by differentiating the k-th row of W. Thus

$$W' = \begin{vmatrix} \phi_1' & \cdots & \phi_n' \\ \phi_1' & \cdots & \phi_n' \\ \phi_1'' & \cdots & \phi_n'' \\ \vdots & & \vdots \\ \phi_1^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} \phi_1 & \cdots & \phi_n \\ \phi_1'' & \cdots & \phi_n'' \\ \phi_1'' & \cdots & \phi_n'' \\ \vdots & & \vdots \\ \phi_1^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix} + \cdots + \begin{vmatrix} \phi_1 & \cdots & \phi_n \\ \phi_1' & \cdots & \phi_n' \\ \phi_1'' & \cdots & \phi_n'' \\ \vdots & & \vdots \\ \phi_1^{(n)} & \cdots & \phi_n^{(n)} \end{vmatrix}$$

The first $n-1$ determinant $V_1, V_2, \cdots, V_{n-1}$ are all zero, since they each have two identical rows. Since $\phi_1, \phi_2, \cdots, \phi_n$ are solutions of $L(y) = 0$ we have

$$\phi_i^{(n)} = -a_1\phi_i^{(n-1)} - \cdots - a_n\phi_i, (i = 1, 2, \cdots, n)$$

and therefore

$$W' = \begin{vmatrix} \phi_1 & \cdots & \phi_n \\ \phi_1' & \cdots & \phi_n' \\ \phi_1'' & \cdots & \phi_n'' \\ \vdots & & \vdots \\ \phi_1^{(n-2)} & \cdots & \phi_n^{(n-2)} \\ -\sum_{j=0}^{n-1} a_{n-j}\phi_1^{(j)} & \cdots & -\sum_{j=0}^{n-j} a_{n-j}\phi_n^{(j)} \end{vmatrix}.$$

The value of this determinant is unchanged if we multiply any row by a number and add to the last row. We multiply the first row by a_n, the second row by $a_{(n-1)}, \cdots$, the $(n-1)-th$ row by a_2, and add these to the last row, obtaining

$$W' = \begin{vmatrix} \phi_1 & \cdots & \phi_n \\ \phi_1' & \cdots & \phi_n' \\ \phi_1'' & \cdots & \phi_n'' \\ \vdots & & \vdots \\ \phi_1^{(n-2)} & \cdots & \phi_n^{(n-2)} \\ -a_1\phi_1^{(n-1)} & \cdots & -a_1\phi_n^{(n-1)} \end{vmatrix} = -a_1 W.$$

Therefore W satisfies the linear first order equation $y' + a_1(x)y = 0$, and thus

$$W(x) = \exp\left[-\int_{x_0}^{x} a_1(t)dt\right] W(x_0).$$

Corollary 3.1 *If the coefficients a_k of L are constants, then*

$$W(\phi_1, \phi_2, \cdots, \phi_n)(x) = e^{-a_1(x-x_0)} W(\phi_1, \phi_2, \cdots, \phi_n)(x_0).$$

Proof:
A consequence of Theorem 3.8 is that n solutions $\phi_1, \phi_2, \cdots, \phi_n$ of

$$L(y) = 0$$

on an interval I are linearly independent there if and only if

$$W(\phi_1, \phi_2, \cdots, \phi_n)(x_0) \neq 0$$

for any particular x_0 in I.

Let us sum up

1. We have proved the properties of the linearly dependent and linearly independent solutions by using Wronskian formula.

2. We have characterized the any linear combination of linearly independent solutions is again a linearly independent solution.

3. Finally, we figured out the Abel's formula.

Check your progress

1. When $r_1 \neq r_2$, find Wronskian $W[e^{r_1 x}, e^{r_2 x}]$?
   (a)$(r_2 - r_1)e^{(r_1+r_2)x_0}$ (b)$(r_1 - r_2)e^{(r_1+r_2)x_0}$
   (c)$(r_2 - r_1)e^{(r_1-r_2)x_0}$ (d)$(r_1 - r_2)e^{(r_1-r_2)x_0}$

2. If ϕ_1 and ϕ_2 are solution of $y'' + x^2 y' + (1-x)y = 0$ such that $y_1(0) = 0$, $y_1'(0) = -1$ and $y_2(0) = -1$, $y_2'(0) = 1$, then the $W(y_1, y_2)$ on R is
   (a) never zero
   (b) identically zero
   (c) zero only at finite number of points
   (d) zero at countable infinite number of points.

3.4 Reduction of the order of a homogeneous equation

Suppose we have found by some means one solution ϕ_1 of the equation

$$L(y) = y^{(n)} + a_1(x)y(n-1) + ... + a_n(x)y = 0.$$

It is then possible to use this information to lower the order of the equation such that it may be solved by one. The idea is the same as in the variation of constants technique. We want to find solutions ϕ to $L(y) = 0$ of the form $\phi = u\phi_1$, where u is any function. If $\phi = u\phi_1$ is the solution, we must have

$$\begin{aligned} 0 &= u\phi_1)^{(n)} + a_1(u\phi_1)^{(n-1)} + \cdots + a_{n-1}(u\phi_1)' + a_n(u\phi_1) \\ &= u^{(n)}\phi_1 + \cdots + u\phi_1^{(n)} + a_1 u^{(n-1)}\phi_1 + \cdots + a_1 u\phi_1^{(n-1)} \\ &\quad + \cdots + a_{n-1}u'\phi_1 + a_{n-1}u\phi_1' + a_n u\phi_1. \end{aligned}$$

The coefficient of u in the above equation is just $L(\phi_1) = 0$. Therefore, if $v = u'$, this is a linear equation of order $n-1$ in v,

$$\phi_1 v^{(n-1)} + \cdots + [n\phi_1^{(n-1)} + a_1(n-1)\phi_1^{(n-2)} + ... + a_{n-1}\phi_1]v = 0. \tag{3.18}$$

The coefficient of $v^{(n-1)}$ is ϕ_1, and hence $\phi_1(x) \neq 0$ on an interval I this equation has $n-1$ linearly independent solutions $v_2, \cdots, v_n$ on I. If x_0 is some point in I, and

$$u_k(x) = \int_{x_0}^{x} v_k(t)dt, \ (k = 2, \cdots, n),$$

then we have $u_k' = v_k$, and the functions

$$\phi_1, u_2\phi_1, \cdots, u_n\phi_1 \tag{3.19}$$

are solutions of $L(y) = 0$. Moreover these functions form a basis for the solutions of $L(y) = 0$ on I. For suppose we have constants $c_1, c_2, \cdots, c_n$ such that

$$c_1\phi_1 + c_2 u_2\phi_1 + \cdots + c_n u_n\phi_1 = 0.$$

Since $\phi_1(x) \neq 0$ on I this implies

$$c_1 + c_2u_2 + \cdots + c_nu_n = 0, \tag{3.20}$$

and differentiating we obtain

$$c_2u_2' + \cdots + c_nu_n' = 0,$$

or

$$c_2v_2 + \cdots + c_nv_n = 0.$$

Since $v_2, \cdots, v_n$ are linearly independent on I we have

$$c_2 = \cdots = c_n = 0,$$

and from 3.20 we obtain $c_1 = 0$ also. Thus the functions in 3.19 form a basis for the solutions of $L(y) = 0$ on I.

Theorem 3.9 *Let ϕ_1 be a solution of $L(y) = 0$ on an interval I, and suppose $\phi_1(x) \neq 0$ on I. If $v_2, \cdots, v_n$ is any basis on I for the solutions of the linear equation 3.18 of order $n-1$, and if*

$$v_k = u_k', \quad (k = 2, \cdots, n)$$

then $\phi_1, u_2\phi_1, \cdots, u_n\phi_1$ is a basis for the solutions of $L(y) = 0$ on I.

Proof:
Given ϕ_1 is the solution of

$$L(y) = y^{(n)} + a_1(x)y^{(n-1)} + ... + a_n(x)y = 0.$$

Now, we find the solution of $L(y) = 0$ of the term $\phi = u\phi_1$, where u is some function.

$$\begin{aligned}
L(u\phi_1) =& (u\phi_1)^n + a_1(u\phi_1)^{(n-1)} + \cdots + a_n(u\phi_1) \\
=& u^n\phi_1 + n_{c_1}u^{(n-1)}\phi_1' + \cdots + u\phi_1^n \\
& + a_1[u^{n-1}\phi_1 + (n-1)_{c_1}u^{(n-2)}\phi_1' + \cdots \\
& + u\phi_1^{(n-1)}] + \cdots + a_{(n-1)}[u'\phi_1 + u\phi_1'] + a_nu\phi_1 \\
=& \phi_1u^n + [n\phi_1^{(1)} + a_1\phi_1]u^{(n-1)} + \cdots + [n\phi_1^{(n-1)} + a_1(n-1)\phi_1^{(n-2)} \\
& + \cdots + a_{n-1}\phi_1]u' + [\phi_1^{(n)} + a_1\phi_1^{(n-1)} + \cdots + a_n\phi_1]u.
\end{aligned}$$

The coefficient of u in the above equation is

$$\phi_1^{(n)} + a_1\phi_1^{(n-1)} + \cdots + a_n\phi_1.$$

Since ϕ_1 is solution of $L(y) = 0$, we have

$$\phi_1^{(n)} + a_1\phi_1^{(n-1)} + \cdots + a_n\phi_1 = 0.$$

If $u' = v$, then

$$\phi_1 v^{n-1} + [n\phi_1^{(1)} + a_1\phi_1]v^{n-2} + \cdots + [n\phi_1^{(n-1} + a_1(n-1)\phi_1^{(n-2)} + \cdots + a_{n-1}\phi_1]v = 0. \quad (3.21)$$

Given: $v_2, \cdots, v_n$ are $(n-1)$ linearly independent solution of $L(y) = 0$. If x_0 is a point in I and

$$u_k = \int_{x_0}^{x} v_k(t)dy.$$

Thus $u_2, \cdots, u_n$ is a solution of equation $L(y) = 0$. Therefore, $\phi_1, u_2\phi_1, \cdots, u_n\phi_1$ is a basis for the solutions of $L(y) = 0$ on I.
Consider the constant

$$c_1\phi_1 + c_2\phi_1 u_2 + \cdots + c_n u_n \phi_1 = 0. \quad (3.22)$$

Given: $\phi_1(x) \neq 0$ on I
By dividing $\phi_1(x)$ on equation 3.22, we get.

$$c_1 + c_2 u_2 + \cdots + c_n u_n = 0. \quad (3.23)$$

Differentiate the above equation with respect to x on both sides, then we have

$$\begin{aligned} c_2u_2' + c_3u_3' + \cdots + c_nu_n' &= 0 \\ \implies c_2v_2 + c_3v_3 + \cdots + c_nv_n &= 0[u_k' = v_k]. \end{aligned}$$

We have that $v_2, \cdots, v_n$ are linearly independent. This implies that

$$c_1 = c_2 = \cdots = c_n = 0,$$

and $\phi_1, u_2\phi_1, \cdots, u_n\phi_1$ are linearly independent. Hence $\phi_1, u_2\phi_1, \cdots, u_n\phi_1$ is a basis for the solutions of $L(y) = 0$ on I.

Theorem 3.10 *If ϕ_1 is a solution of*

$$L(y) = y'' + a_1(x)y' + a_2(x)y = 0 \quad (3.24)$$

on an interval I, and $\phi_1(x) \neq 0$ on I, a second solution ϕ_2 of 3.24 on I is given by

$$\phi_2(x) = \phi_1(x) \int_{x_0}^{x} \frac{1}{[\phi_1(s)]^2} \exp\left[-\int_{x_0}^{x} a_1(t)dt\right]. \quad (3.25)$$

The functions ϕ_1, ϕ_2 form a basis for the solutions of 3.24 on I.

Proof:
Let ϕ_1 be a solution of $L(y) = 0$. Then we have $L(\phi_1) = 0$, that is,

$$\phi_1'' + a_1(x)\phi_1' + a_2(x)\phi_1 = 0. \quad (3.26)$$

Our aim is to find a solution ϕ_2 of $L(y) = 0$. There is $\phi_2 = u\phi_1$ where u is a function. Since ϕ_2 is a solution of $L(y) = 0$, then $L(\phi_2) = 0$. Therefore,

$$\begin{aligned} \phi_2'' + a_1(x)\phi_2' + a_2(x)\phi_2 &= 0 \\ (u\phi_1)'' + a_1(x)(u\phi_1)' + a_2(x)(u\phi_1) &= 0 \end{aligned}$$

$$[u''\phi_1 + u'\phi_1' + u\phi_1''] + a_1(x)[u'\phi_1 + u\phi_1'] + a_2(x)(u\phi_1) = 0$$
$$u''\phi_1 + [2\phi_1' + a_1(x)\phi_1']u' + [\phi_1'' + a_1(x)\phi_1' + a_2(x)\phi_1]u = 0.$$

Then by the equation 3.26 we have,

$$u''\phi_1 + [2\phi_1' + a_1(x)\phi_1']u' = 0. \tag{3.27}$$

Take $u' = v$. Then

$$\phi_1 v' + [2\phi_1' + a_1(x)\phi_1]v = 0$$
$$\phi_1 v' + 2\phi_1' v + a_1(x)\phi_1 v = 0$$

Multiply the above equation by ϕ_1

$$\implies \phi_1^2 v' + 2\phi_1\phi_1' v + a_1(x)\phi_1^2 v = 0$$
$$\implies (\phi_1^2 v)' + a_1(x)(\phi_1^2 v) = 0. \tag{3.28}$$

If we take $\phi_2 v = y$, then it becomes

$$y' + a_1(x)y = 0.$$

Therefore

$$(\phi_1^2 v)e^{\int_{x_0}^{x} a_1(x)dx} = \int 0 e^{\int_{x_0}^{x} a_1(x)dx} dx + c$$
$$(\phi_1^2 v)e^{\int_{x_0}^{x} a_1(x)dx} = c.$$

Thus, we have

$$(\phi_1^2 v) = ce^{-\int_{x_0}^{x} a_1(x)dx},$$

where x_0 is a point in I and c is constant. Since any constant multiple of a solution is again a solution, we get

$$[\phi_1^2(x)]v(x) = e^{-\int_{x_0}^{x} a_1(x)dx}$$
$$v(x) = \frac{1}{[\phi_1(x)]^2} e^{-\int_{x_0}^{x} a_1(x)dx}$$

is a solution of equation 3.24,

$$u(x) = \int_{x_0}^{x} \frac{1}{[\phi_1(s)]^2} e^{-\int_{x_0}^{s} a_1(x)dx} ds.$$

Thus

$$\phi_2(x) = \phi_1(x) \int_{x_0}^{x} \frac{1}{[\phi_1(s)]^2} e^{-\int_{x_0}^{s} a_1(x)dx} ds.$$

Since $L(y) = y'' + a_1(x)y' + a_2(x)y = 0$ has two linearly independent solutions on I. Hence ϕ_1, ϕ_2 form a basis for the solutions of $L(y) = y'' + a_1(x)y' + a_2(x)y = 0$ on I.

Example 3.1 *Find the basis for the solutions of the equation* $y'' - \frac{2}{x^2}y = 0$ *on* $0 < x < \infty$.

Solution:
It is clear that the function ϕ_1 given by $\phi_1(x) = x^2$ is a solution on $0 < x < \infty$, and since this function does not vanish on this interval there is another independent solution ϕ_1 of the form $\phi_2 = u\phi_1$. If $v = u'0$ we find that v satisfies

$$x^2v' + 4xy = 0, \text{ or } xy' + 4v = 0.$$

A solution for this is given by

$$v(x) = x^{-4}, \quad (0 < x < \infty),$$

and therefore a choice for u is

$$u(x) = -\frac{1}{3x^2} \quad (0 < x < \infty).$$

This implies

$$\phi_1(x) = -\frac{1}{3x}, \quad (0 < x < \infty),$$

but since any constant times a solution is a solution, we may as well choose for a second solution $\phi_2(x) = x^{-1}$. Hence x^2, x^{-1} form a basis for the solutions on $(0 < x < \infty)$.

Let us sum up

1. We have characterized the homogeneous equation of order n.
2. We have defined the basis of homogeneous equation of order n.
3. We have rectified the properties of the reduction of the order of a homogeneous equation.
4. Finally, we solved some illustrative examples.

Check your progress

5. If $\phi_1(x) = x$ is a solution of $x^2y'' - xy' + y = 0$ for $x > 0$, then the second solution is
   (a) $\phi_2(x) = x^{-1}$ (b) $\phi_2(x) = xlogx$ (c) $\phi_2(x) = x^2$ (d) $\phi_2(x) = x$
6. Two solutions ϕ of $x^3y''' - 3xy' + 3y = 0$, $x > 0$ are $\phi_1(x) = x, \phi_2(x) = x^2$. Find the third Independent solution.
   (a) $\phi_3(x) = x^{-1}$ (b) $\phi_3(x) = x^2$ (c) $\phi_3(x) = x^3$ (d) $\phi_3(x) = x^{-2}$

3.5 Homogeneous equations with analytic coefficients

If g is a function defined on an interval I containing a point x_0, we say that g is analytic at x_0 if g can be expanded in a power series about x_0 which has a positive radius of convergence. Thus g is analytic at x_0 if it can be represented in the form

$$g(x) = \sum_{k=0}^{\infty} c_k(x - x_0)^k, \tag{3.29}$$

where the c_k are constants, and the series converges for $|x - x_0| < r_0, r_0 > 0$, all of its derivatives exist on $|x - x_0| < r_0$, and they may be computed by differentiating the series term by term. Thus, for example

$$g'(x) = \sum_{k=1}^{\infty} k c_k (x - x_0)^{k-1},$$

and

$$g''(x) = \sum_{k=2}^{\infty} k(k-1) c_k (x - x_0)^{k-2},$$

and the differentiated series converge on $|x - x_0| < r_0$ also.

Theorem 3.11 *(Existence Theorem for Analytic Coefficients) Let x_0 be a real number, and suppose that the coefficients $a_1, \cdots, a_n$ in*

$$L(y) = y^{(n)} + a_1(x) y^{(n-1)} + \cdots + a_n(x) y$$

have convergent power series expansions in power of $x - x_0$ on an interval

$$|x - x_0| < r_0, \quad r_0 > 0.$$

If $\alpha_1, \cdots, \alpha_n$ are any n constants, then there exists a solution ϕ of the problem

$$L(y) = 0, y(x_0) = \alpha_1, \cdots, y^{(n-1)}(x_0) = \alpha_n,$$

with a power series expansion

$$\phi(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \tag{3.30}$$

convergent for $|x - x_0| < r_0$. We have

$$k! c_k = \alpha_{k+1}, (k = 0, 1, \cdots, n-1),$$

and c_k for $k \geq n$ may be computed in terms of $c_0, c_1, \cdots, c_{n-1}$ by substituting the series 3.58 into $L(y) = 0$.

Proof:
Let us consider the two power series,

$$\sum_{k=0}^{\infty} c_k x^k, \quad \sum_{k=0}^{\infty} C_k x^k,$$

$$|c_k| \leq C_k, \quad C_k \geq 0, (k = 0, 1, \cdots),$$

and that the series

$$\sum_{k=0}^{\infty} C_k x^k$$

converges on $|x| < r$, for some $r > 0$. Then the series

$$\sum_{k=0}^{\infty} c_k x^k$$

also converges for $|x| < r$. This is usually called the *comparison test* for convergence. The second result we require is that if a series

$$\sum_{k=0}^{\infty} \alpha_k x^k \tag{3.31}$$

is convergent for $|x| < r_0$, then for any x, $|x| = r < r_0$, there is a constant $M > 0$ such that

$$r^k |\alpha_k| \leq M, (k = 0, 1, \cdots). \tag{3.32}$$

This is not difficult to show. Since the series 3.31 is convergent for $|x| = r$ its terms must tend to zero,

$$|\alpha_k x^k| = |\alpha_k| r^k \to 0, \quad (k \to \infty).$$

In particular there is an integer $N > 0$ such that

$$|\alpha_k| r^k \leq 1, \quad (k > N).$$

Let M be the largest number among

$$|\alpha_0|, |\alpha_1| r, \cdots, |\alpha_N| r^N, 1.$$

Then clearly 3.32 is valid for this M. We now consider the equation

$$L(y) = y'' + a_1(x)y' + b(x)y = 0, \tag{3.33}$$

where a, b are functions having expansions

$$a(x) = \sum_{k=0}^{\infty} \alpha_k x^k, \quad b(x) = \sum_{k=0}^{\infty} \beta_k x^k, \tag{3.34}$$

which converge for $|x| < r_0$ for some $r_0 > 0$. Given any constants α_1, α_2 we want to produce a solution ϕ of 3.33 satisfying

$$\phi_0 = \alpha_1, \quad \phi'(0) = \alpha_2,$$

and which can be written in the form

$$\phi(x) = \sum_{k=0}^{\infty} c_k x^k, \tag{3.35}$$

where the series converges for $|x| < r_0$. If this series is convergent we must have

$$c_0 = \alpha_1,$$

and the constants $c_k(k \leq 2)$ must satisfy a relation, which we now compute. We have

$$\phi'(x) = \sum_{k=0}^{\infty}(k+1)c_{k+1}x^k,$$

and

$$\phi'(x) = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k. \tag{3.36}$$

Now we obtain

$$\begin{aligned} a(x)\phi'(x) &= \left(\sum_{k=0}^{\infty}\alpha_k x^k\right)\left(\sum_{k=0}^{\infty}(k+1)c_{k+1}x^k\right) \\ a(x)\phi'(x) &= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\alpha_{k-j}(j+1)c_{j+1}\right)x^k, \end{aligned} \tag{3.37}$$

and

$$\begin{aligned} b(x)\phi(x) &= \left(\sum_{k=0}^{\infty}\beta_k x^k\right)\left(\sum_{k=0}^{\infty}c_k x^k\right) \\ &= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\beta_{k-j}c_j\right)x^k. \end{aligned} \tag{3.38}$$

Adding the above equations we get

$$L(\phi)(x) = \sum_{k=0}^{\infty}\left[(k+1)(k+2)c_{k+2} + \sum_{j=0}^{k}\alpha_{k-j}(j+1)c_{j+1} + \sum \beta_{k-j}c_j\right]x^k = 0.$$

Thus the c_k must satisfy

$$(k+2)(k+1)c_{k+2} = -\left[\sum_{j=0}^{k}\alpha_{k-j}(j+1)c_{j+1} + \sum \beta_{k-j}c_j\right], \tag{3.39}$$

$$(k = 0, 1, 2\cdots).$$

It is enough to show that if the c_k, for $k \leq 2$, are defined by 3.36, then the series

$$\sum_{k=0}^{\infty}c_k x^k \tag{3.40}$$

is convergent for $|x| < r_0$. To prove this we can use of the two results concerning power series we mentioned earlier. Let r be any number satisfying $0 < r < r_0$. Since the series in 3.31 are convergent for $|x| = r$ we have a constant $M > 0$ such that

$$|\alpha_j|\, r^j \leq M, \quad |\beta_j|\, r^j \leqq M, \quad (j = 0, 1, 2, \cdots).$$

Using this in 3.39 we find that

$$(k+2)(k+1)\,|c_{k+2}| \;\leq\; \frac{M}{r^k}\sum_{j=0}^{k}\left[(j+1)\,|c_{i+1}| + |c_i|\right]r^j$$

$$\leq \quad \frac{M}{r^k}\sum_{j=0}^{k}\left[(j+1)\left|c_{j+1}\right|+\left|c_j\right|\right]r^j + M\left|c_{k+1}\right|r. \tag{3.41}$$

Now let us define

$$C_0 = |c_0|, \quad C_1 = |c_1|$$

and C_k for $k \geq 2$ by

$$(k+2)(k+1)C_{k+2} = \frac{M}{r^k}\sum_{j=0}^{k}\left[(j+1)C_{j+1}+C_j\right]r^j + MC_{k+1}r, \tag{3.42}$$

$(k = 0, 1, 2, \cdots)$. Comparing 3.41 with 3.42 we have

$$|c_k| \leqq C_k, \quad C_k \geqq 0, \quad (k = 0, 1, 2, \cdots). \tag{3.43}$$

Now, we have to find for which x the series

$$\sum_{k=0}^{\infty} C_k x^k \tag{3.44}$$

is convergent. We find that

$$(k+1)kC_{k+1} = \frac{M}{r^{k-1}}\sum_{j=0}^{k-1}\left[(j+1)C_{j+1}+C_i\right]r^j + MC_k r$$

and

$$k(k-1)C_k = \frac{M}{r^{k-1}}\sum_{j=1}^{k}\left[(j+1)C_{j+1}+C_i\right]r^j + MC_{k-1}r_1$$

for large k. From these expressions we obtain

$$\begin{aligned} r(k+1)kC_{k+1} &= \frac{M}{r^{k-2}}\sum_{j=0}^{k-2}\left[(j+1)C_{j+1}+C_j\right]r^j + M\left[kC_k + C_{k-1}\right]r + MC_k r^2 \\ &= k(k-1)C_k - MC_{k-1}r_3 + MkC_k r + MC_{k-1}r + MC_k r^2 \\ &= \left[k(k-1) + Mkr + Mr^2\right]C_k. \end{aligned} \tag{3.45}$$

Hence

$$\left|\frac{C_{k+1}x^{k+1}}{C_k x^k}\right| = \frac{\left[k(k-1)+Mkr+Mr^2\right]}{r(k+1)k}|x|,$$

which tends to ∞.

Thus, by the ratio test, the series 3.40 converges for $|x| < r$, and since r was any number satisfying $0 < r < r_0$, we have shown at last that the series 3.40 converges for $|x| < r_0$.

Let us sum up

1. We have characterized the homogeneous equation with analytic coefficients with examples.

2. We have discussed the existence theorem for analytic coefficients.
3. Also, we defined the comparison test for convergence for proving the existence result.

Check your progess

7. State Existence Theorem for Analytic Coefficients.

3.6 The Legendre equation

Some of the important differential equations met in physical problems are second order linear equation with analytic coefficients. One of these is the Legendre equation

$$L(y) = (1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0, \tag{3.46}$$

where α is a constant.
Dividing 3.46 by $(1-x^2)$, we obtain the standard form of given equation as

$$y'' - \frac{2x}{1-x^2}y' + \frac{\alpha(\alpha+1)}{1-x^2}y = 0. \tag{3.47}$$

The coefficients of the resulting equations

$$\begin{aligned} a_1(x) =& \frac{-2x}{1-x^2}, \\ a_2(x) =& \frac{\alpha(\alpha+1)}{1-x^2}, \end{aligned}$$

are analytic at $x = 0$. Indeed,

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + \cdots = \sum_{k=0}^{\infty} x^{2k},$$

and this series converges for $|x| < 1$. Thus a_1 and a_2 have the series expansions

$$\begin{aligned} a_1(x) =& \sum_{k=0}^{\infty}(-2)x^{2k+1}, \\ a_2(x) =& \sum_{k=0}^{\infty}\alpha(\alpha+1)x^{2k}, \end{aligned}$$

which converge for $|x| < 1$. From Theorem 3. 12 it follows that the solutions of $L(y) = 0$ on $|x| < 1$ have convergent power series expansion there. We proceed to find a basis for these solutions.

Let ϕ be any solution of the Legendre equation on $|x| < 1$, and suppose

$$\phi(x) = c_0 + c_1 x + c_2 x^2 + \cdots = \sum_{k=0}^{\infty} c_k x^k. \tag{3.48}$$

We have

$$\phi'(x) = c_1 + 2c_2x + 3c_3x^2 + \cdots = \sum_{k=0}^{\infty} kc_kx^{k-1},$$

$$-2x\phi'(x) = \sum_{k=0}^{\infty} -2kc_kx^k,$$

$$\phi''(x) = 2c_2 + 3 \cdot 2c_3x + \cdots = \sum_{k=0}^{\infty} k(k-1)c_kx^{k-2},$$

$$-x^2\phi''(x) = \sum_{k=0}^{\infty} -k(k-1)c_kx^k.$$

Note that $\phi''(x)$ may also be written as

$$\phi''(x) = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k, \tag{3.49}$$

Since ϕ is a solution of $L(y) = 0$, we get

$$(1-x^2)\phi''(x) - 2x\phi'(x) + \alpha(\alpha+1)\phi(x) = 0. \tag{3.50}$$

Substitute $\phi(x), \phi'(x), \phi''(x)$ values in the above equation, we get

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=0}^{\infty} -k(k-1)c_kx^k - 2\sum_{k=0}^{\infty} kc_kx^k + \alpha(\alpha+1)\sum_{k=0}^{\infty} c_kx^k = 0$$

$$\sum_{k=0}^{\infty}[(k+2)(k+1)c_{k+2} - k(k-1)c_k - 2kc_k + \alpha(\alpha+1)c_k]x^k = 0,$$

$$\sum_{k=0}^{\infty}[(k+2)(k+1)c_{k+2} + (\alpha+k+1)(\alpha-k)c_k]x^k = 0.$$

We must have all the coefficients of the powers of x equal to zero. Hence

$$(k+2)(k+1)c_{k+2} + (\alpha+k+1)(\alpha-k)c_k = 0, \qquad (k = 0, 1, 2, \cdots)$$

$$(k+2)(k+1)c_{k+2} = -(\alpha+k+1)(\alpha-k)c_k. \tag{3.51}$$

This is the recursion relation which gives c_{k+2} in terms of c_k. For $k = 0$ we obtain

$$c_2 = -\frac{(\alpha+1)\alpha}{2}c_0,$$

and for $k = 1$ we get,

$$c_3 = -\frac{(\alpha+2)(\alpha-1)}{3 \cdot 2}c_1.$$

Similarly, letting k = 2, 3 in 3.51 we obtain

$$c_4 = -\frac{(\alpha+3)(\alpha-2)}{4 \cdot 3}c_2 = \frac{(\alpha+3)(\alpha+1)\alpha(\alpha-2)}{4 \cdot 3 \cdot 2}c_0,$$

$$c_5 = -\frac{(\alpha+4)(\alpha-3)}{5\cdot 4}c_3 = \frac{(\alpha+4)(\alpha+2)(\alpha-1)(\alpha-3)}{5\cdot 4\cdot 3\cdot 2}c_1.$$

The pattern now becomes clear, and it follows by induction that for $m = 1, 2, \cdots$

$$c_{2m} = (-1)^m \frac{(\alpha+2m-1)(\alpha+2m-3)\cdots(\alpha+1)\alpha(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!}c_0,$$

$$c_{2m+1} = (-1)^m \frac{(\alpha+2m)(\alpha+2m-2)\cdots(\alpha+2)(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)!}c_1.$$

All coefficient are determined in terms of $c_0$and c_1, and we must have

$$\phi(x) = c_0\phi_1(x) + c_1\phi_2(x),$$

where

$$\phi_1(x) = 1 - \frac{(\alpha+1)\alpha}{2!}x^2 + \frac{(\alpha+3)(\alpha+1)\alpha(\alpha-2)}{4!}x^4 - \cdots,$$

or

$$\phi_1(x) = 1+\sum_{m=1}^{\infty}(-1)^m \frac{(\alpha+2m-1)(\alpha+2m-3)\ldots(\alpha+1)\alpha(\alpha-2)\cdots(\alpha-2m+2)}{(2m)!}x^{2m}, \tag{3.52}$$

and

$$\phi_2(x) = x - \frac{(\alpha+2)(\alpha-1)}{3!}x^2 + \frac{(\alpha+4)(\alpha+2)(\alpha-1)(\alpha-3)}{5!}x^3 - \cdots,$$

or

$$\phi_2(x) = x+\sum_{m=1}^{\infty}(-1)^m \frac{(\alpha+2m)(\alpha+2m-2)\cdots(\alpha+2)(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{(2m+1)!}x^{2m+1}. \tag{3.53}$$

Both ϕ_1 and ϕ_2 are solutions of the Legendre equation, those corresponding to the choices

$$c_0 = 1, c_1 = 0, \text{ and } c_0 = 0, c_1 = 1,$$

respectively. They form a basis for the solutions, since

$$\begin{aligned} \phi_1(0) &= 1, & \phi_2(0) &= 0, \\ \phi_1'(0) &= 0, & \phi_2'(0) &= 1. \end{aligned}$$

We notice that if α is a non-negative even integer

$$n = 2m, \quad (m = 0, 1, 2, \cdots),$$

then ϕ_1 has only a finite number of non-zero terms. Indeed, in this case ϕ_1 is a polynomial of degree n containing only even powers of x. For example,

$$\begin{aligned} \phi_1(x) &= 1, \quad (\alpha = 0), \\ \phi_1(x) &= 1 - 3x^3, \quad (\alpha = 2), \end{aligned}$$

$$\phi_1(x) = 1 - 10x^2 + \frac{35}{8}x^4, \quad (\alpha = 4).$$

The solution ϕ_2 is not a polynomial in this case since none of the coefficients in the series 3.53 vanish.

A similar situation occurs when α is a positive odd integer n. Then ϕ_2 is a polynomial of degree n having only odd powers of x, and ϕ_1 is not a polynomial. For example

$$\begin{aligned}
\phi_1(x) =& x, \quad (\alpha = 1), \\
\phi_2(x) =& x - \frac{5}{3}x^3, \quad (\alpha = 3), \\
\phi_2(x) =& x - \frac{14}{8}x^3 + \frac{21}{5}x^5, \quad (\alpha = 5).
\end{aligned}$$

We consider in more detail these polynomial solutions when $\alpha = n$, a non-negative integer. The polynomial solution P_n of degree n of

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0, \tag{3.54}$$

satisfying $P_n(1) = 1$ is called the n-th Legendre polynomial. In order to justify this definition we must show that there is just one such solution for each non-negative integer n. This will be established by way of a slight detour, which is of interest in itself.

Let ϕ be the polynomial of degree n defined by

$$\phi(x) = \frac{d^n}{dx^n}(x^2 - 1)^n.$$

This ϕ satisfies the Legendre equation 3.54. Indeed, let

$$u(x) = (x^2 - 1)^n.$$

Then we obtain by differentiating

$$(x^2 - 1)u' - 2nxu = 0.$$

Differentiating this expression $n + 1$ times yields

$$(x^2 - 1)u^{(n+2)} + 2x(n+1)u^{(n+1)} + (n+1)nu^{(n)} - 2nxu^{(n+1)} - 2n(n+1)u^{(n)} = 0.$$

Since $\phi = u^{(n)}$ we obtain

$$(1 - x^2)\phi''(x) - 2x\phi'(x) + n(n+1)\phi(x) = 0,$$

and we have shown that ϕ satisfies 3.54.

This polynomial ϕ satisfies

$$\phi(1) = 2^n n!.$$

This can be seen by noting that

$$\phi(x) = [(x^2 - 1)^n]^{(n)} = [(x - 1)^n (x + 1)^n]^{(n)}$$

$$= [(x-1)^n]^{(n)}(x+1)^n + \text{ terms with } (x-1) \text{ as a factor}$$
$$= n!(x+1)^n + \text{ terms with } (x-1) \text{ as a factor}.$$

Hence $\phi(1) = n!2^n$, as stated.

It is now clear that the function P_n given by

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \tag{3.55}$$

is the n-th Legendre polynomial, provided we can show that there is no other polynomial solution of 3.54 which is 1 at $x = 1$.

Suppose ψ is any polynomial solution of 3.54. Then for some constant c we must have $\psi = c\phi_1$ or $\psi = c\phi_2$, according as n is even or odd. Here ϕ_1, ϕ_2 are the solutions of 3.54, 3.55. Suppose n is even, for example. Then, for $|x| < 1$,

$$\psi = c\phi_1 + d\phi_2$$

for some constants c, d, since ϕ_1, ϕ_2 form a basis for the solutions on $|x| < 1$. But then $\psi - c\phi_1$ is a polynomial, whereas $d\phi_2$ is not a polynomial in case $d \neq 0$. Hence $d = 0$. In particular the function P_n given by 3.55 satisfies $P_n = c\phi_1$ for some constant c, if n is even. Since $1 = P_n(1) = c\phi_1(1)$, we see that $\phi_1(1) \neq 0$. A similar result is valid if n is odd. Thus no non-trivial polynomial solution of the Legendre equation can be zero at $x = 1$. From this it follows that there is only one polynomial P_n satisfying 3.54 and $P_n(1) = 1$, for if $\tilde{P}_n$ was another, then $P_n - \tilde{P}_n$ would be a polynomial solution, and $P_n(1) - \tilde{P}_n(1) = 0$.

The first few Legendre polynomials are

$$P_0(x) = 1,\quad P_1(x) = x,\quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2},$$
$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x,\quad P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}.$$

Let us sum up

1. We have discussed the Legendre's equations and functions used for solving differential equations..

2. We have rectified the solutions of the Legendre equation with suitable examples.

Check your Progress

8. The value of the Legendre polynomial $P_2(x)$ is
   (a) $\frac{3}{2}x - 1$ (b) $\frac{3}{2}x^2 - \frac{1}{2}x^3$ (c) $\frac{3}{2}x^2 - \frac{1}{2}$ (d) $\frac{3}{2} - \frac{1}{2}x^2$

9. The value of Legendre polynomial $P_3(x)$ is
   (a) $\frac{5}{2}x^2 - \frac{3}{2}$ (b) $\frac{5}{2}x^3 - \frac{3}{2}x$ (c) $\frac{5}{2}x^2 + \frac{3}{2}$ (d) $\frac{5}{2}x^3 + \frac{3}{2}x$

10. The n-th Legendre polynomials $P_n(x)$ is given by
    (a) $P_n(x) = -\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$ (b) $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2+1)^n$
    (c) $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^3-1)^n$ (d) $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$

Summary

This unit discusses linear differential equations with variable coefficients. The topics include:

- Linear Independence and Wronskian: Understanding linear independence of solutions using the Wronskian.
- Reduction of Order: A technique used when one solution of a second-order ODE is known.
  - Given one solution $\phi_1(x)$, a second solution $\phi_2(x) = v(x)\phi_1(x)$ can be found.
- Series Solutions: Used when variable coefficients prevent simple analytic solutions. Methods like Frobenius allow solutions near singular points.
- The Legendre's differential equation can be solved in series of ascending or descending power of x. The solution in descending powers of x is more important than the one in ascending powers.
- The solution of Legendre's equation is called Legendre's function.

Glossary

- *Basis*: A basis is a set of linearly independent vectors that span a vector space, meaning any vector in the space can be expressed as a combination of these basis vectors.
- *Power series*: A power series is a way of expressing a function as an infinite sum of terms that involve powers of a variable. Each term in the series has a constant coefficient and a certain power of the variable, allowing functions to be approximated or studied around a specific point.
- *Legendre equation*: The Legendre equation is a type of differential equation that arises in problems with spherical symmetry, such as gravitational or electric fields. Its solutions, known as Legendre polynomials.
- *Abel's formula*: Abel's formula, also known as Abel's identity, relates the Wronskian of two solutions of a second-order linear differential equation to the coefficients of the equation. It shows that the Wronskian either remains constant or varies exponentially, helping to analyze the behavior of solutions.

Self-assesment questions

1. Let $P_n(x)$ be the Legendre polynomial of degree n and let $P_{m+1}(0) = \frac{-m}{m+1} p_{m+1}(0), m = 1, 2, \cdots$. If $P_n(0) = \frac{-5}{16}$, then $\int_{-1}^{1} P_n^2(x)dx =$,
   (a) $\frac{2}{13}$ (b) $\frac{2}{9}$ (c) $\frac{5}{16}$ (d) $\frac{2}{5}$
2. Using the fact that $P_0(x) = 1$ is a solution of
   $$(1 - x^2)y'' - 2xy' = 0,$$
   find a second independent solution by the method of Sec. 3.5.

3. Verify that the function Q_1 defined by

$$Q_1(x) = \frac{x}{2}\log((1+x)/(1-x)) - 1(|x| < 1),$$

is a solution of the Legendre equation when $\alpha = 1$.

4. Show that $P_n(-1) = (-1)^n$.

EXERCISES

1. Consider the equation

$$L(y) = y'' + a_1(x)y' + a_2(x)y = 0,$$

where a_1, a_2 are continuous on some interval I. Let ϕ_1, ϕ_2 and ψ_1, ψ_2 be two bases for the solutions of $L(y) = 0$. Show that there is a non-zero constant k such that

$$W(\psi_1, \psi_2)(x) = kW(\phi_1, \phi_2)(x).$$

2. Consider the same equation as in Ex. 1. Show that a_1 and a_2 are uniquely determined by any basis ϕ_1, ϕ_2 for the solutions of $L(y) = 0$ (Hint: Try solving for a_1, a_2 from the equations

$$L(\phi_1) = 0, \quad L(\phi_2) = 0$$

Show that

$$a_1 = -\frac{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1'' & \phi_2'' \end{vmatrix}}{W(\phi_1, \phi_2)}, \quad a_2 = \frac{\begin{vmatrix} \phi_1' & \phi_2' \\ \phi_1'' & \phi_2'' \end{vmatrix}}{W(\phi_1, \phi_2)}.)$$

3. Consider the equation

$$y'' + \alpha(x)y = 0,$$

where α is a continuous function on $-\infty < x < \infty$ which is of period $\xi > 0$. Let ϕ_1, ϕ_2 be the basis for the solutions satisfying

$$\begin{aligned} \phi_1(0) &= 1, \quad & \phi_2(0) &= 0, \\ \phi_1'(0) &= 0, \quad & \phi_2'(0) &= 1. \end{aligned}$$

(a) Show that $W(\phi_1, \phi_2)(x) = 1$ for all x.

(b) Show that there is at least one non-trivial solution ϕ of period ξ if, and only if,

$$\phi_1(\xi) + \phi_2'(\xi) = 2$$

(c) Show that there exists a non-trivial solution ϕ satisfying

$$\phi(x + \xi) = -\phi(x)$$

if, and only if,

$$\phi_1(\xi) + \phi_2'(\xi) = -2$$

(Hint: Show that such a ϕ exists if, and only if,

$$\phi(\xi) = -\phi(0) and \phi'(\xi) = -\phi'(0)$$

See Ex. 6, Sec. 3.)

(d) If $\phi_1(\xi) + \phi_2'(\xi) = -2$ show that there exists a non-trivial solution of period 2ξ. (Hint: Use (c). Alternately, use (b) with ξ replaced by ξ.)

4. A differential equation and a function ϕ_1 are given in each of the following. Verify that the function ϕ_1 satisfies the equation, and find a second independent solution,

   (a) $x^2y'' - 7xy' + 15y = 0,\ \phi_1(x) = x^3, (x > 0)$.

   (b) $y'' - 4xy' + (4x^2 - 2)y = 0,\ \phi_1(x) = e^{x^2}$.

   (c) $xy'' - (x+1)y' + y = 0,\ \phi_1(x) = e^x, (x > 0)$.

   (d) $(1 - x^3)y'' - 2xy' + 2y = 0,\ \phi_1(x) = x, (0 < x < 1)$.

   (e) $y'' - 2xy' + 2y = 0,\ \phi_1(x) = x, (x > 0)$.

5. One solution of
$$x^3y''' - 3x^2y'' + 6xy' - 6y = 0$$
for $x > 0$ is $\phi_1(x) = x$. Find a basis for the solutions for $x > 0$.

6. Find two linearly independent power series solutions (in powers of x) of the following equations:

   (a) $y'' - xy' + y = 0$

   (b) $y'' + 3x^2y' - xy = 0$

   (c) $y'' - x^2y + 0$

   (d) $y'' + x^3y' + x^2y = 0$

   (e) $y'' + y = 0$.

   For what values of x do the series converge?

7. Find the solution ϕ of
$$y'' + (x-1)^2y' - (x-1)y = 0$$
in the form
$$\phi(x) = \sum_{k=0}^{\infty} c_k(x-1)^k,$$
which satisfies $\phi(1) = 1,\ \phi'(1) = 0$. (Hint: Let $x - 1 = \xi$.)

8. Find the solution ϕ of
$$(1 + x^2)y'' + y = 0$$
of the form
$$\phi(x) = \sum_{k=0}^{\infty} c_k x^k,$$
which satisfies $\phi(0) = 0, \phi'(0) = 1$. (Note: When the equation is written in the form
$$y'' + \frac{1}{1+x^2}y = 0,$$

it is one with analytic coefficients at $x = 0$, since

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^2 + \cdots = \sum_{k=0}^{\infty}(-1)^k x^{2k},$$

which converges for $|x| < 1$. However to compute ϕ it is best to substitute the series for ϕ directly into the given equation.) What is the largest $r > 0$ such that the series for ϕ converges for $|x| < r$?

9. The equation

$$y'' + e^x y = 0$$

has a solution ϕ of the form

$$\phi(x) = \sum_{k=0}^{\infty} c_k x^k$$

which satisfies $\phi(0) = 1, \phi'(0) = 0$. Compute $c_0, c_1, c_2, c_3, c_4, c_5$. (Hint: $c_k = \frac{\phi^k(0)}{k!}$ and $\phi''(x) = -e^x\phi(x)$.)

10. Compute the solution ϕ of

$$y''' - xy = 0$$

which satisfies $\phi(0) = 1, \phi'(0) = 0, \phi''(0) = 0$.

11. The equation

$$y'' - 2xy' + 2\alpha y = 0,$$

where α is a constant, is called the Hermite equation.

(a) Find two linearly independent solutions on $-\infty < x < \infty$.

(b) Show that there is a polynomial solution of degree n, in case $\alpha = n$ non-negative integer.

(c) Show that the polynomial H_n defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

is a solution of the Hermite equation in case $\alpha = n$ is a non-negative integer. This solution H_n, is called the n-th Hermile polynomial. (Hint: If $u(x) = e^{-x^2}$ show that $u'(x) + 2xu(x) = 0$. Differentiate this equation n times to obtain

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0 \tag{3.56}$$

$n \geq 1$. Differentiate H_n to obtain

$$H_n'(x) = 2xH_n(x) - H_{n+1}(x) \tag{3.57}$$

for $n \geq 0$. Use 3.56 and 3.57 to show H_n is a solution of the Hermite equation.)

(d) Computs H_0, H_1, H_2, H_3.

12. Show that the coefficient of x^n in $P(x)$ is

$$\frac{(2n)!}{2^n(n!)^2}.$$

13. Show that

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0, (n \neq m).$$

(Hint: Note that

$$[(1-x^2)P_n']' = -n(n+1)P_n,$$
$$[(1-x^2)P_m']' = -m(m+1)P_m.$$

Hence

$$\begin{aligned} P_m[(1-x^2)P_n']' - P_n[(1-x^2)P_m']' &= \{(1-x^2)[P_mP_n' - P_m'P_n]\}' \\ &= [m(m+1) - n(n+1)]P_mP_n. \end{aligned}$$

Integrate from -1 to 1.)

14. Show that

$$\int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}$$

(Hint: Let $u(x) = (x^2-1)^n$. Then from 3.55

$$P_n(x) = \frac{1}{2^n n!}u^{(n)}(x).$$

Show that $u^{(k)}(1) = u^{(k)}(-1) = 0$ if $0 \leq k < n$. Then, integrating by parts,

$$\begin{aligned} \int_{-1}^{1} u^{(n)}(x)u^{(n)}(x)dx &= u^{(n)}(x)u^{(n-1)}(x)\Big|_{-1}^{1} - \int u^{(n+1)}(x)u^{(n-1)}(x)dx \\ &= -\int_{-1}^{1} u^{(n+1)}(x)u^{(n-1)}(x)dx \\ &= \cdots = (-1)^n \int_{-1}^{1} u^{(2n)}(x)u(x)dx. \\ &= (2n)! \int_{-1}^{1} (1-x^2)^n dx. \end{aligned}$$

To compute the latter integral let $x = \sin\theta$, and obtain

$$\int_{-1}^{1}(1-x^2)^n dx = 2\int_{0}^{\frac{\pi}{2}} \cos^{(2n+1)}\theta d\theta = \frac{2(2^n n!)^2}{(2n+1)!}.)$$

Answer for check your progress

1. Existence Theorem: Let $a_1, a_2, \cdots, a_n$ be continuous functions on an interval I containing the point x_0. If $\alpha_1, \alpha_2, \cdots, \alpha_n$ are any n constants, there exists a solution ϕ of

$$L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0$$

on I satisfying

$$\phi(x_0) = \alpha_1, \phi^{(1)}(x_0) = \alpha_2, \cdots, \phi^{(n-1)}(x_0) = \alpha_n.$$

2. Uniqueness Theorem: Let x_0 be in I, and let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be any n constants. There is at most one solution ϕ of $L(y) = 0$ on I satisfying

$$\phi(x_0) = \alpha_1, \phi'(x_0) = \alpha_2, \cdots, \phi^{(n-1)}(x_0) = \alpha_n.$$

3. (a) 4. (a) 5. (b) 6. (c)
7. Existence Theorem for Analytic Coefficients: Let x_0 be a real number, and suppose that the coefficients $a_1, \cdots, a_n$ in

$$L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y$$

have convergent power series expansions in power of $x - x_0$ on an interval

$$|x - x_0| < r_0, \quad r_0 > 0.$$

If $\alpha_1, \cdots, \alpha_n$ are any n constants, then there exists a solution ϕ of the problem

$$L(y) = 0, y(x_0) = \alpha_1, \cdots, y^{(n-1)}(x_0) = \alpha_n,$$

with a power series expansion

$$\phi(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \tag{3.58}$$

convergent for $|x - x_0| < r_0$. We have

$$k!c_k = \alpha_{k+1}, (k = 0, 1, \cdots, n-1),$$

and c_k for $k \geq n$ may be computed in terms of $c_0, c_1, \cdots, c_{n-1}$ by substituting the series 3.58 into $L(y) = 0$.
8. (c) 9. (b) 10. (d)

Suggested Readings

1. E. A. Coddington and N. Levinson. Theory of Ordinary Differential. Equations. New Delhi: Tata Mc Graw-Hill, 1972.

2. G. F. Simmons, Differential Equations with Applications and Historical Notes, Tata McGraw Hill, New Delhi, 1974.

3. N. N. Lebedev, Special Functions and Their Applications, Prentice Hall of India, New Delhi, 1965.

Unit 4

Linear Equations with Regular Singular Points

OBJECTIVE:
Following completion of this unit, you will be capable of understanding the concept of linear equations with regular singular points of homogeneous equations and the less simple exceptional case. Further, we explain regular and irregular singularities. And we discuss the significance of the Frobenius method and analyze the general solution of Bessel's equation. Finally, we understand how Bessel's equation is integrated for $n = 0$ and explain the concept of recurrence formula for $J_\alpha(x)$.

4.1 Introduction

In this unit we investigate linear equations with variable coefficients

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0. \tag{4.1}$$

We shall assume that the coefficients $a_0, a_1, \cdots, a_n$ are analytic at some point x_0, and we shall be interested in an important case when $a_0(x_0) = 0$.

Definition 4.1 *A point x_0 such that $a_0(x_0) = 0$ is called a singular point of the equation 4.1.*

Definition 4.2 *We say that the point x_0 is a regular singular points for equation 4.1 if the equation can be written in the form*

$$(x - x_0)^n y^{(n)} + b_1(x - x_0)^{(n-1)} y^{(n-1)} + \cdots + b_n(x)y = 0. \tag{4.2}$$

near x_0, where the functions $b_1, b_2, \cdots, b_n$ are analytic at x_0.

If the functions $b_1, b_2, \cdots, b_n$ can be written in the form

$$b_k(x) = (x - x_0)^k \beta_k(x), \qquad (k = 1, \cdots, n),$$

where $\beta_1, \cdots, \beta_n$ are analytic at x_0, we see that 4.2 becomes

$$y^{(n)} + \beta_1(x)y^{(n-1)} + \cdots + \beta_n(x)y = 0 \tag{4.3}$$

upon dividing out $(x-x_0)^n$. Thus 4.2 is a generalization of the equation with analytic coefficients.

An equation of the form

$$c_0(x)(x-x_0)^n y^{(n)} + c_1(x)(x-x_0)^{(n-1)} y^{(n-1)} + \cdots + c_n(x)y = 0$$

has a regular singular point at x_0 if $c_0, c_1, \cdots, c_n$ are analytic at x_0, and $c_0(x_0) \neq 0$. This is because we may divide by $c_0(x)$, for x near x_0, to obtain an equation of the form 4.2 with $b_k(x) = \dfrac{c_k(x)}{c_0(x)}$, and it can be shown that these b_k are analytic at x_0.

We first consider the simplest case of an equation, not of the type 4.3, having a regular singular point. This is the Euler equation, which is the case of 4.2 with $b_1, b_2, \cdots, b_n$ all constants. For $x > x_0$ such solutions ϕ turn out to be of the form

$$\phi(x) = (x-x_0)^r \sigma(x) + (x-x_0)^s \rho(x) \log(x-x_0),$$

where r, s are constants, and σ, ρ are analytic at x_0. The method used is to show that the coefficients of the series for the analytic functions σ, ρ can be computed in a recursive fashion, and then the series obtained actually converge near the singular point.

Example 4.1 *Consider the equation*

$$x^2 y'' - y' - \frac{3}{4} y = 0. \tag{4.4}$$

The origin $x_0 = 0$ is a singular point, but not a regular singular point since the coefficient -1 of y' is not of the form $xb_1(x)$, where b_1 is analytic at 0. We may formally solve this equation by series

$$\sum_{k=0}^{\infty} c_k x^k, \tag{4.5}$$

where the coefficients c_k satisfy the recursion formula

$$(k+1)c_{k+1} = \left(k^2 - k - \frac{3}{4}\right) c_k, \qquad (k = 0, 1, 2, \cdots). \tag{4.6}$$

If $c_0 \neq 0$, the ratio test applied to 4.5, 4.6, shows that

$$\left| \frac{c_{k+1} x^{k+1}}{c_k x^k} \right| = \left| \frac{k^2 - k - \frac{3}{4}}{k+1} \right| |x| \to \infty, \tag{4.7}$$

as $k \to \infty$, provided $|x| \neq 0$.Thus the series 4.5 will only converge for $x = 0$, and therefore does not represent a function near $x = 0$, much less a solution of 4.4.

4.2 The Euler equation

A second order equation having a regular singular point at the origin is the Euler equation

$$L(y) = x^2 y'' + axy' + by = 0,$$

where a, b are constants.We first consider this equation for $x > 0$, and observe that the coefficient of $y^{(k)}$ in $L(y)$ is a constant times x^k. If r is any constant, x^r has the property that its k-th derivative times x^k is a constant times x^r.

Theorem 4.1 *Consider the second order Euler equation*

$$x^2y'' + axy' + by = 0 \quad (a, b \text{ constants}),$$

and the polynomial q given by

$$q(r) = r(r-1) + ar + b.$$

A basis for the solutions of the Euler equation on any interval not containing $x = 0$ is given by

$$\phi_1(x) = |x|^{r_1}, \;\; \phi_2(x) = |x|^{r_2},$$

in case r_1, r_2 are distinct roots of q, and by

$$\phi_1(x) = |x|^{r_1}, \;\; \phi_2(x) = |x|^{r_1}\log|x|,$$

if r_1 is a root of q of multiplicity two.

Proof:
Let us consider the second order Euler equation having a regular singular point at the origin

$$L(y) = x^2y'' + axy' + by = 0, \tag{4.8}$$

where a, b are constants.

We first consider this equation for $x > 0$, and observe that the coefficient of $y^{(k)}$ in $L(y)$ is a constant times x^k. If r is any constant, x^r has the property that its k-th derivative times x^k is a constant times x^r. This suggests trying for a solution of $L(y) = 0$ a power of x. Let $y = x^r$. Then

$$y' = rx^{r-1}, \quad y'' = r(r-1)x^{r-2},$$

and

$$\begin{aligned} L(x^r) &= x^2r(r-1)x^{r-2} + axrx^{r-1} + bx^r \\ &= r(r-1)x^r + arx^r + bx^r \\ &= (r(r-1) + ar + b)x^r \\ L(x^r) &= q(r)x^r, \end{aligned} \tag{4.9}$$

where $q(r) = r(r-1) + ar + b$.

It is clear that if r_1 is a root of q, then

$$L(x^{r_1}) = 0.$$

Thus the function ϕ_1 given by $\phi_1(x) = x^{r_1}$ is a solution of 4.8 for $x > 0$.
Case(i): If r_2 is the other root of q, and $r_1 \neq r_2$, we obtain another solution ϕ_2 given by $\phi_2(x) = x^{r_2}$.
Case(ii): If the roots $r_1 = r_2$(repeated roots), we obtain

$$q(r_1) = 0, \;\; q'(r_1) = 0.$$

Differentiating $L(x^r) = q(r)x^r$ with respect to r, we get

$$\begin{aligned}
\frac{\partial}{\partial r}L(x^r) &= L\left(\frac{\partial}{\partial r}x^r\right) \\
&= L\left(\frac{\partial}{\partial r}e^{\log x^r}\right) = L\left(\frac{\partial}{\partial r}e^{r\log x}\right) \\
&= L\left(e^{r\log x}\log x\right) = L\left(e^{\log x^r}\log x\right) \\
&= L\left(x^r\log x\right) \\
&= x^2(x^r\log x)'' + ax(x^r\log x)' + b(x^r\log x). \qquad (4.10)
\end{aligned}$$

Now

$$\begin{aligned}
(x^r\log x)'' &= \left(rx^{r-1}\log x + x^r\frac{1}{x}\right)' = \left(rx^{r-1}\log x + x^{r-1}\right)' \\
&= r(r-1)x^{r-2}\log x + rx^{r-1}\frac{1}{x} + (r-1)x^{r-2} \\
&= r(r-1)x^{r-2}\log x + rx^{r-2} + (r-1)x^{r-2} \\
&= x^{r-2}(r-1)[r\log x + 1] + rx^{r-2}. \qquad (4.11)
\end{aligned}$$

Substitute the value of 4.10 in 4.11, we get

$$\begin{aligned}
\frac{\partial}{\partial r}L(x^r) &= x^2\left[x^{r-2}(r-1)[r\log x+1] + rx^{r-2}\right] + ax\left[rx^{r-1}\log x + x^{r-1}\right] + b(x^r\log x) \\
&= (1+r\log x)(r-1)x^r + rx^r + a\left[x^r(r\log x+1)\right] + b(x^r\log x) \\
&= \left[(1+r\log x)(r-1) + r + a(r\log x+1) + b\log x\right]x^r \\
&= \left[(r-1) + r(r-1)\log x + r + ar\log x + a + b\log x\right]x^r \\
&= \left[(r(r-1)+ar+b)\log x + r - 1 + r + a\right]x^r \\
&= \left[q(r)\log x + 2r - 1 + a\right]x^r \\
&= \left[q(r)\log x + q'(r)\right]x^r
\end{aligned}$$

and if $r = r_1$ is a repeated root of $q(r)$, we get $q(r_1) = 0, q'(r_1) = 0$. Therefore

$$\frac{\partial}{\partial r}[L(x^{r_1})] = L[x^{r_1}\log x] = 0.$$

Therefore $\phi_2 = x^{r_2}\log x$ is a second solution associated with the root r_1. Thus $\phi_1 = x^{r_1}$, $\phi_2 = x^{r_1}\log x$ are solutions of the given equation.

In either case the solutions ϕ_1, ϕ_2 are linearly independent for $x > 0$.

Case(i):

If $r_1 \neq r_2$, and c_1, c_2 are constants such that

$$c_1x^{r_1} + c_2x^{r_2} = 0, \quad (x > 0),$$

then

$$c_1 + c_2x^{r_2-r_1} = 0, \quad (x > 0). \qquad (4.12)$$

Differentiate with respect to x, we get

$$0 + (r_2 - r_1)c_2x^{r_2-r_1-1} = 0$$

$$\Rightarrow c_2 = 0,$$

and from 4.12 we obtain $c_1 = 0$ also.
Case(ii):
If $r_1 = r_2$, and c_1, c_2 are constants such that

$$c_1 x^{r_1} + c_2 x^{r_1} \log x = 0, \quad (x > 0),$$

Dividing the above equation by x^{r_1}, then

$$c_1 + c_2 \log x = 0, \quad (x > 0). \tag{4.13}$$

Differentiating with respect to x, we get

$$0 + c_2 \frac{1}{x} = 0$$
$$\Rightarrow c_2 = 0,$$

and from 4.13 we obtain $c_1 = 0$ also.

We define x^r for r complex by

$$x^r = e^{r \log x}, \quad (x > 0).$$

Then we have

$$(x^r)' = r(\log x)' e^{r \log x} = r x^{-1} x^r = r x^{r-1},$$

and

$$\frac{\partial}{\partial r}(x^r) = \frac{\partial}{\partial r}(e^{r \log x}) = (\log x) e^{r \log x} = x^r \log x,$$

which are the formulas we used in the calculations.
Now we have to find the solutions of 4.8 for the case $x < 0$ also. In this case consider $(-x)^r$, where r is a constant. Then we have for $x < 0$

$$y = [(-x)^r] \Rightarrow y' = r(-x)^{r-1} \Rightarrow y'' = r(r-1)(-x)^{r-2},$$

and hence

$$x[(-x)^r]' = r(-x)^r, \;\; x^2[(-x)^r]'' = r(r-1)(-x)^r.$$

Thus

$$\begin{aligned} L[(-x)^r] &= x^2 r(r-1)(-x)^{r-2} + ax(-r)(-x)^{r-1} + b(-x)^r \\ &= r(r-1)(-x)^r + ar(-x)^r + b(-x)^r \\ &= (r(r-1) + ar + b)(-x)^r \\ &= q(r)(-x)^r, \qquad\qquad (x < 0), \end{aligned} \tag{4.14}$$

where $q(r)$ is the polynomial defined by $q(r) = r(r-1) + ar + b$.
Thus, $L[(-x)^r] = 0$ if $q(r) = 0$. (i.e)$(-x)^r$ is a solution of $L(y) = 0$ if and only if r is a root of the polynomial $q(r)$.

Now, $q(r) = r^2 + (a-1)r + b$ is a second degree equation. By fundamental theorem of algebra, $q(r)$ has two roots r_1 and r_2.

Case(i):
If $r_1 \neq r_2$, then $\phi_1(x) = (-x)^{r_1},\ \ \phi_2(x) = (-x)^{r_2}$ are the distinct independent solutions of $L(y) = 0$ for $x < 0$.
Case(ii):
If $r_1 = r_2$, clearly $(-x)^{r_1}$ is a solution of $L(y) = 0$. Since r_1 is a repeated root of $q(r)$, we have $q(r_1) = 0, q'(r_1) = 0$. Now

$$\begin{aligned}
\frac{\partial}{\partial r}(L(-x)^r) &= L\left(\frac{\partial}{\partial r}(-x)^r\right) \\
&= L\left(\frac{\partial}{\partial r}e^{\log(-x)^r}\right) = L\left(\frac{\partial}{\partial r}e^{r\log(-x)}\right) \\
&= L\left(e^{r\log(-x)}\log(-x)\right) = L\left(e^{\log(-x)^r}\log(-x)\right) \\
&= L\left[(-x)^r\log(-x)\right] \qquad (4.15)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial r}\left[q(r)(-x)^r\right] &= \frac{\partial}{\partial r}\left[q(r)e^{\log(-x)^r}\right] \\
&= \frac{\partial}{\partial r}\left[q(r)e^{r\log(-x)}\right] \\
&= q(r)e^{r\log(-x)}\log(-x) + q'(r)e^{r\log(-x)} \\
&= q(r)(-x)^r\log(-x) + q'(r)(-x)^r \\
&= \left[q(r)\log(-x) + q'(r)\right](-x)^r. \qquad (4.16)
\end{aligned}$$

Now equating the equations 4.15 and 4.16, we get

$$L\left[(-x)^r\log(-x)\right] = \left[q(r)\log(-x) + q'(r)\right](-x)^r.$$

If $r = r_1$

$$L\left[(-x)^{r_1}\log(-x)\right] = \left[q(r_1)\log(-x) + q'(r_1)\right](-x)^{r_1}.$$

This implies that $(-x)^{r_1}\log(-x)$ is a solution of $L(y) = 0$. Thus $\phi_1(x) = (-x)^{r_1}$ and $\phi_2(x) = (-x)^{r_1}\log(-x)$ are the solutions of $L(y) = 0$.
For $x > 0$,

$$\begin{aligned}
&\phi_1(x) = x^{r_1},\ \ \phi_2(x) = x^{r_2},\quad \text{if } r_1 \neq r_2, \\
&\phi_1(x) = x^{r_1},\ \ \phi_2(x) = x^{r_1}\log x,\quad \text{if } r_1 = r_2.
\end{aligned}$$

For $x < 0$,

$$\begin{aligned}
&\phi_1(x) = (-x)^{r_1},\ \ \phi_2(x) = (-x)^{r_2},\quad \text{if } r_1 \neq r_2, \\
&\phi_1(x) = (-x)^{r_1},\ \ \phi_2(x) = (-x)^{r_1}\log(-x),\quad \text{if } r_1 = r_2.
\end{aligned}$$

Since $|x| = x$ for $x > 0$, and $|x| = -x$ for $x < 0$, we can write the solutions for any $x \neq 0$
in the following way:

$$\phi_1(x) = |x|^{r_1},\ \ \phi_2(x) = |x|^{r_2},\ \ (x \neq 0)$$

in case $r_1 \neq r_2$, and

$$\phi_1(x) = |x|^{r_1},\ \ \phi_2(x) = |x|^{r_1}\log|x|,\ \ (x \neq 0)$$

in case $r_1 = r_2$.

Example 4.2 *Find a basis for the the solutions of the equation*

$$x^2y'' + xy' + y = 0$$

for the case $x \neq 0$.

Solution:
Given $x^2y'' + xy' + y = 0$. The polynomial q is given by

$$\begin{aligned} q(r) &= r(r-1) + r + 1 \\ &= r^2 - r + r + 1 \\ &= r^2 + 1. \end{aligned}$$

Its roots are $r_1 = i, r_2 = -i$. Thus a basis for the the solutions is given by

$$\phi_1(x) = |x|^i, \qquad \phi_2(x) = |x|^{-i}, \quad (x \neq 0),$$

where we have $|x|^i = e^{i \log |x|}$.

Note that in this case another basis ψ_1, ψ_2 is given by

$$\psi_1(x) = \cos(\log |x|), \quad \psi_2(x) = \sin(\log |x|), \ (x \neq 0).$$

Theorem 4.2 *Let $r_1, r_2, \cdots, r_s$ be the distinct roots of the indicial polynomial q for*

$$L(y) = x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_n y = 0, \tag{4.17}$$

and suppose r_1 has multiplicity m_1. Then the n functions

$$\begin{gathered} |x|^{r_1}, |x|^{r_1} \log |x|, \cdots, |x|^{r_1} \log^{m_1 - 1} |x|; \\ |x|^{r_2}, |x|^{r_2} \log |x|, \cdots, |x|^{r_2} \log^{m_2 - 1} |x|; \\ \vdots \\ |x|^{r_s}, |x|^{r_s} \log |x|, \cdots, |x|^{r_s} \log^{m_s - 1} |x| \end{gathered}$$

form a basis for the solution of the n-th order Euler equation 4.17 on any interval not containing $x = 0$.

Proof:
For any constant r, we have

$$\begin{aligned} y &= |x|^r, \\ y' &= (|x|^r)' = r|x|^{r-1} \\ y'' &= (|x|^r)'' = r(r-1)|x|^{r-2} \\ &\vdots \\ y^k &= (|x|^r)^k = r(r-1)\cdots(r-k+1)|x|^{r-k} \\ x^{(k)}(|x|^r)^k &= r(r-1)\cdots(r-k+1)|x|^r. \end{aligned}$$

Then

$$L(|x|^r) = x^k r(r-1)\cdots(r-k+1)|x|^{r-k} + a_1 x^{k-1} r(r-1)\cdots(r-k)|x|^{r-k-1} + \cdots + a_k |x|^r$$

$$
\begin{aligned}
&= r(r-1)\cdots(r-k+1)|x|^r + a_1 r(r-1)\cdots(r-k)|x|^r + \cdots + a_k|x|^r \\
&= (r(r-1)\cdots(r-k+1) + a_1 r(r-1)\cdots(r-k) + \cdots + a_k)|x|^r \\
&= q(r)|x|^r, \qquad (4.18)
\end{aligned}
$$

where q is the polynomial of degree n defined by

$$q(r) = r(r-1)\cdots(r-n+1) + a_1 r(r-1)\cdots(r-n+2) + \cdots + a_n.$$

This polynomial is called the indicial polynomial for the Euler equation 4.17. We obtain

$$
\begin{aligned}
\frac{\partial^k}{\partial r^k}(L|x|^r) &= L\left(\frac{\partial^k}{\partial r^k}|x|^r\right) \\
&= L\left(\frac{\partial^k}{\partial r^k}e^{\log|x|^r}\right) = L\left(\frac{\partial^k}{\partial r^k}e^{r\log|x|}\right) \\
&= L\left(e^{r\log|x|}\log|x|\right) = L\left(e^{\log|x|^r}\log|x|\right) \\
&= L\left(|x|^r\log|x|\right) \\
\frac{\partial^k}{\partial r^k}(L|x|^r) &= \Big[q^k(r) + kq^{(k-1)}(r)\log|x| \\
&\quad + \frac{k(k-1)}{2!}q^(k-2)(r)\log^2|x| + \cdots + q(r)\log^k|x|\Big]|x|^r. \qquad (4.19)
\end{aligned}
$$

If r_1 is a root of q of multiplicity m_1, then

$$q(r_1) = 0, q'(r_1) = 0, \cdots q^{m_1-1}(r_1) = 0,$$

and we see that $|x|^{r_1}, |x|^{r_1}\log|x|, \cdots, |x|^{r_1}\log^{m_1-1}|x|$ are the solutions of $L(y) = 0$. Repeating the process for each root of q we obtain that the result

$$|x|^{r_2}, |x|^{r_2}\log|x|, \cdots, |x|^{r_2}\log^{m_2-1}|x|; \cdots; |x|^{r_s}, |x|^{r_s}\log|x|, \cdots, |x|^{r_s}\log^{m_s-1}|x|$$

form a basis for the solutions of the n-th order Euler equation for any interval not containing $x \neq 0$.

Let us sum up

1. We have defined a second-order equation having a regular singular point as the Euler equation.

2. Finally, we solved some illustrative examples.

Check your progress

1. The solution of the differential equation $x^2y'' + xy' + 4y = 0$ for $|x| > 0$ is given by,
   (a) $\phi(x) = c_1|x|^{2i} + c_2|x|^{-2i}$ (b) $\phi(x) = c_1|x|^{i} + c_2|x|^{2i}$
   (c) $\phi(x) = c_1|x|^{-i} + c_2|x|^{-2i}$ (d) None of these.

2. The solution of differential equation $x^2y'' + xy' - 4y = -x$ for $x > 0$ is given by,
   (a) $\phi(x) = c_1x^2 + c_2x^{-2}$ (b) $\phi(x) = c_1x^{-3} + c_2x^2$
   (c) $\phi(x) = c_1x^{-3} + c_2x^{-2}$ (d) None of these

4.3 Second order equations with regular singular points-an example

The second order equation with regular singular point at x_0 has the form

$$(x - x_0)^2 y'' + a(x)(x - x_0)y' + b(x)y = 0 \tag{4.20}$$

where $a(x), b(x)$ are analytic at x_0. Thus a, b has the power series expansions

$$a(x) = \sum_{k=0}^{\infty} \alpha_k (x - x_0)^k, \qquad b(x) = \sum_{k=0}^{\infty} \beta_k (x - x_0)^k,$$

which are convergent on some interval $|x - x_0| < r_0$, for some $r_0 > 0$.

Let $t = x - x_0$. Then $x = x_0 + t$, and

$$\begin{aligned}\tilde{a}(t) = a(x_0 + t) &= \sum_{k=0}^{\infty} \alpha_k (x_0 + t - x_0)^k \\ &= \sum_{k=0}^{\infty} \alpha_k (t)^k\end{aligned}$$

and

$$\begin{aligned}\tilde{b}(t) = b(x_0 + t) &= \sum_{k=0}^{\infty} \beta_k (x_0 + t - x_0)^k \\ &= \sum_{k=0}^{\infty} \beta_k (t)^k.\end{aligned}$$

The power series for $\tilde{a}, \tilde{b}$ converge on the interval $|t| < r_0$ about $t = 0$. Let ϕ be any solution of 4.20, and define $\tilde{\phi}$ by

$$\tilde{\phi}(t) = \tilde{\phi}(x_0 + t).$$

Then

$$\begin{aligned}\frac{d\tilde{\phi}}{dt}(t) &= \frac{d\phi}{dx}(x_0 + t), \\ \frac{d^2\tilde{\phi}}{dt^2}(t) &= \frac{d^2\phi}{dx^2}(x_0 + t),\end{aligned}$$

and

$$\begin{aligned}(x - x_0)^2 \phi''(x) + a(x)(x - x_0)\phi' + b(x)\phi(x) = 0 \\ t^2 \frac{d^2\tilde{\phi}}{dt^2}(t) + t\tilde{a}(t)\frac{d\tilde{\phi}}{dt}(t) + \tilde{b}(t)\tilde{\phi}(t) = 0.\end{aligned}$$

Thus ϕ satisfies

$$t^2 u'' + \tilde{a}(t) t u' + \tilde{b}(t) u = 0, \tag{4.21}$$

where now $u' = du/dt$. This is an equation with a regular singular point at $t = 0$.

Conversely, if $\tilde{\phi}$ satisfies 4.21 the function ϕ given by $\phi(x) = \tilde{\phi}(x - x_0)$ satisfies 4.20. In this sense 4.21 is equivalent to 4.20.

With $x_0 = 0$ in 4.20 we may write 4.20 as

$$L(y) = x^2 y'' + a(x) x y' + b(x) y = 0, \tag{4.22}$$

where a, b are analytic at the origin, and have power series expansions

$$a(x) = \sum_{k=0}^{\infty} \alpha_k x^k, \qquad b(x) = \sum_{k=0}^{\infty} \beta_k x^k, \tag{4.23}$$

which are convergent on an interval $|x| < r_0, r_0 > 0$. The Euler equation is the special case of 4.22 with the constants a, b.

Example 4.3 *Find the solutions of the equation*

$$L(y) = x^2 y'' + \frac{3}{2} x y' + x y = 0, \tag{4.24}$$

which has a regular singular point at the origin.

Solution:
Let us restrict our attention to $x > 0$. Since it is not an Euler equation, we can not expect it to have a solution of the form x^r there. However we try for a solution

$$\begin{aligned} \phi(x) &= x^r \sum_{k=0}^{\infty} c_k x^k = x^r (c_0 + c_1 x + + c_1 x^2 + \cdots,) \\ &= c_0 x^r + c_1 x^{r+1} + c_2 x^{r+2} + \cdots, \qquad (c_0 \neq 0), \end{aligned} \tag{4.25}$$

that is, x^r times a power series. We operate formally and see what conditions must be satisfied by r and $c_0, c_1, c_2, \cdots$ in order that this ϕ be a solution of 4.24. Computing we find that

$$\begin{aligned} \phi'(x) &= c_0 r x^{r-1} + c_1 (r+1) x^r + c_2 (r+2) x^{r+1} + \cdots, \\ \phi''(x) &= c_0 r (r-1) x^{r-2} + c_1 (r+1) r x^{r-1} + c_2 (r+2)(r+1) x^r + \cdots, \end{aligned}$$

and hence

$$\begin{aligned} x^2 \phi''(x) &= c_0 r (r-1) x^r + c_1 (r+1) r x^{r+1} + c_2 (r+2)(r+1) x^{r+2} + \cdots, \\ \frac{3}{2} x \phi'(x) &= \frac{3}{2} c_0 r x^r + \frac{3}{2} c_1 (r+1) x^{r+1} + \frac{3}{2} c_2 (r+2) x^{r+2} + \cdots, \\ x \phi(x) &= c_0 x^{r+1} + c_1 x^{r+2} + c_2 x^{r+3} + \cdots. \end{aligned}$$

Adding the above equations, we obtain

$$\begin{aligned} (i.e) \quad L(\phi)(x) &= x^2 \phi''(x) + x \phi'(x) + x \phi(x), \\ L(\phi)(x) &= \left[r(r-1) + \frac{3}{2} r \right] c_0 x^r + \left\{ \left[(r+1) r + \frac{3}{2}(r+1) \right] c_1 + c_0 \right\} x^{r+1} \end{aligned}$$

$$+\left\{\left[(r+2)(r+1)+\frac{3}{2}(r+2)\right]c_2+c_1\right\}x^{r+2}+\cdots.$$

If we let

$$q(r)=r(r-1)+\frac{3}{2}r=r\left(r+\frac{1}{2}\right),$$

this can be written as

$$\begin{aligned}L(\phi)(x)&=q(r)c_0x^r+[q(r+1)c_1+c_0]x^{r+1}+[q(r+2)c_2+c_1]x^{r+2}+\cdots\\&=q(r)c_0x^r+x^r\sum_{k=1}^{\infty}[q(r+k)c_k+c_{k-1}]x^k.\end{aligned}$$

If ϕ is to satisfy $L(\phi)(x)=0$ all coefficients of the powers of x must vanish. Since we assumed $c_0\neq 0$ this implies

$$\begin{aligned}q(r)&=0,\\q(r+k)c_k+c_{k-1}&=0,\qquad (k=1,2,\cdots).\end{aligned}\tag{4.26}$$

The polynomial q is called the indicial polynomial for 4.24. It is the coefficient of the lowest power of x appearing in $L(\phi)(x)$, and from 4.26 we see that its roots are the only permissible values of r for which there are solutions of the form 4.25. In our example these roots are

$$r_1=0,\ r_2=-\frac{1}{2}.$$

The second set of equations in 4.26 delimits $c_1,c_2,\cdots$ in terms of c_0 and r. If $q(r+k)\neq 0$ for $k=1,2,\cdots$, then

$$\begin{aligned}q(r+k)c_k+c_{k-1}&=0\\q(r+k)c_k&=-c_{k-1}\\c_k&=-\frac{c_{k-1}}{q(r+k)},\qquad (k=1,2,\cdots,).\end{aligned}$$

Substituting the k values in the above equation, we obtain

$$\begin{aligned}c_1&=(-1)\frac{c_0}{q(r+1)}\\c_2&=\frac{(-1)c_1}{q(r+2)}=\frac{(-1)}{q(r+2)}\frac{(-1)c_0}{q(r+1)}\\&=\frac{(-1)^2c_0}{q(r+2)q(r+1)}\end{aligned}$$

and

$$c_k=\frac{(-1)^kc_0}{q(r+k)q(r+k-1)\cdots q(r+1)},\qquad (k=1,2,\cdots).$$

If $r_1=0$,

$$q(r_1+k)=q(k)\neq 0\ \text{ for }\ k=1,2,\cdots,$$

since the other root of q is $r_2 = -\frac{1}{2}$. Similarly if $r_2 = -\frac{1}{2}$,

$$q(r_2 + k) = q\left(-\frac{1}{2} + k\right) \neq 0 \quad \text{for} \quad k = 1, 2, \cdots.$$

Letting $c_0 = 1$ and $r = r_1 = 0$ we obtain, at least formally, a solution ϕ_1 given by

$$\begin{aligned}
\phi(x) &= c_0 x^0 + c_1 x + c_2 x^2 + \cdots \\
\phi(x) &= c_0 + \sum_{k=1}^{\infty} c_k x^k \\
\phi_1(x) &= c_0 + \sum_{k=1}^{\infty} \frac{(-1)^k c_0 x^k}{q(k)q(k-1)\cdots q(1)}, \\
&= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{q(k)q(k-1)\cdots q(1)}, \quad (\because\ c_0 = 1),
\end{aligned}$$

and letting $c_0 = 1$ and $r = r_2 = -\frac{1}{2}$ we obtain another solution

$$\begin{aligned}
c_k &= \frac{(-1)^k c_0}{q(-\frac{1}{2} + k)q(-\frac{1}{2} + k - 1)\cdots q(-\frac{1}{2} + 1)} \\
&= \frac{(-1)^k x^k}{q(k - \frac{1}{2})q(k - \frac{3}{2})\cdots q(\frac{1}{2})} \\
\phi_2(x) &= c_0 x^{-\frac{1}{2}} + c_1 x^{-\frac{1}{2}+1} + c_2 x^{-\frac{1}{2}+2} + \cdots
\end{aligned}$$

$$\begin{aligned}
&= c_0 x^{-\frac{1}{2}} + \sum_{k=1}^{\infty} c_k x^{-\frac{1}{2}+k} \\
&= c_0 x^{-\frac{1}{2}} + \sum_{k=1}^{\infty} \frac{(-1)^k c_0 x^{-\frac{1}{2}+k}}{q(k - \frac{1}{2})q(k - \frac{3}{2})\cdots q(\frac{1}{2})} \\
\phi_2(x) &= x^{-\frac{1}{2}} + x^{-\frac{1}{2}} \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{q(k - \frac{1}{2})q(k - \frac{3}{2})\cdots q(\frac{1}{2})}, \quad (\because\ c_0 = 1).
\end{aligned}$$

These functions ϕ_1, ϕ_2 will be solutions provided the series converge on some interval containing $x = 0$. Let us write the series for ϕ_1 in the form

$$\phi_1(x) = \sum_{k=0}^{\infty} d_k(x).$$

Using the ratio test we obtain

$$\begin{aligned}
\frac{d_{k+1}(x)}{d_k(x)} &= \frac{(-1)^{k+1} x^{k+1}}{q(k+1)q(k)\cdots q(1)} \frac{(-1)^k x^k}{q(k)q(k-1)\cdots q(1)} = \frac{(-1)(x)}{q(k+1)} \\
\left|\frac{d_{k+1}(x)}{d_k(x)}\right| &= \frac{|x|}{|q(k+1)|} = \frac{|x|}{(k+1)(k+\frac{3}{2})} \to 0
\end{aligned}$$

as $k \to \infty$, provided $|x| < \infty$. Thus the series defining ϕ_1 is convergent for all finite x. The same can be shown to hold for the series multiplying $x^{-\frac{1}{2}}$ in the expression for ϕ_2. Thus ϕ_1, ϕ_2 are solutions of 4.24 for all $x > 0$.

To obtain solutions for $x < 0$ we note that all the above computations go through if x^r is replaced everywhere by $|x|^r$, where

$$|x|^r = e^{r \log |x|}. \tag{4.27}$$

Thus two solutions of 4.24 which are valid for all $x \neq 0$ are given by

$$\phi_1(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{q(k)q(k-1)\cdots q(1)},$$

and

$$\phi_2(x) = |x|^{-\frac{1}{2}} \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^k}{q(k-\frac{1}{2})q(k-\frac{3}{2})\cdots q(\frac{1}{2})} \right].$$

Note:

1. The definition 4.27 implies that $|x|^{\frac{1}{2}}$ is the positive square root of $|x|$.

2. The above example illustrate the general fact that an equation 4.22 with regular singular point at the origin always has a solution ϕ of the form

$$\phi_{(x)} = |x|^r \sum_{k=0}^{\infty} c_k x^k, \tag{4.28}$$

   where r is a constant, and the series converges on the interval $|x| < r_0$. Moreover r, and the constants c_k, may be computed by substituting 4.28 into the differential equation.

4.3.1 Second order equation with regular singular points - the general case

Theorem 4.3 *Consider the equation*

$$x^2 y'' + a(x) x y' + b(x) y = 0,$$

where a, b have convergent power series expansions for

$$|x| < r_0, \qquad r_0 > 0.$$

Let $r_1, r_2 (Re\ r_1 \geq Re\ r_2)$ be the roots of the indicial polynomial

$$q(r) = r(r-1) + a(0)r + b(0).$$

For $0 < |x| < r_0$ there is a solution ϕ_1 of the form

$$\phi_1(x) = |x|^{r_1} \sum_{k=0}^{\infty} c_k x^k, \quad (c_0 = 1),$$

where the series converges for $|x| < r_0$. If $r_1 - r_2$ is not zero or a positive integer, there is a second solution ϕ_2 for $0 < |x| < r_0$ of the form

$$\phi_2(x) = |x|^{r_2} \sum_{k=0}^{\infty} \tilde{c}_k x^k, \qquad (\tilde{c}_0 = 1),$$

where the series converges for $|x| < r_0$.

The coefficients $c_k, \tilde{c}_k$ can be obtained by substitution of the solutions into the differential equation.

Proof:
Suppose we have a solution ϕ of the form

$$\begin{aligned} \phi(x) &= x^r \sum_{k=0}^{\infty} c_k x^k \\ &= \sum_{k=0}^{\infty} c_k x^{(k+r)}, \qquad (c_0 \neq 0) \end{aligned} \tag{4.29}$$

for the equation

$$x^2 y'' + a(x) x y' + b(x) y = 0, \tag{4.30}$$

where

$$a(x) = \sum_{k=0}^{\infty} \alpha_k x^k, \qquad b(x) = \sum_{k=0}^{\infty} \beta_k x^k, \tag{4.31}$$

for $|x| < r_0$. Then

$$\begin{aligned} \phi'(x) &= \sum_{k=0}^{\infty} c_k (k+r) x^{k+r-1} \\ &= x^{r-1} \sum_{k=0}^{\infty} c_k (k+r) x^k, \\ \phi''(x) &= \sum_{k=0}^{\infty} c_k (k+r)(k+r-1) x^{k+r-2} \\ &= x^{r-2} \sum_{k=0}^{\infty} c_k (k+r)(k+r-1) x^k, \end{aligned}$$

and hence

$$\begin{aligned} b(x)\phi(x) &= \left(\sum_{k=0}^{\infty} \beta_k x^k \right) \left(x^r \sum_{k=0}^{\infty} c_k x^k \right) \\ &= x^r \sum_{k=0}^{\infty} \tilde{\beta}_k (x)^k, \end{aligned}$$

where $\tilde{\beta}_k = \sum_{j=0}^{k} c_j \beta_{k-j}$,

$$\begin{aligned}
xa(x)\phi'(x) &= x\left(\sum_{k=0}^{\infty}\alpha_k(x)^k\right)\left(x^{r-1}\sum_{k=0}^{\infty}c_k(k+r)x^k\right)\\
&= x^r\left(\sum_{k=0}^{\infty}\alpha_k(x)^k\right)\left(\sum_{k=0}^{\infty}c_k(k+r)x^k\right)\\
&= x^r\sum_{k=0}^{\infty}\tilde{\alpha}_k(x)^k
\end{aligned}$$

where $\tilde{\alpha}_k = \sum_{j=0}^{k}(j+r)c_j\alpha_{k-j}$,

$$\begin{aligned}
x^2\phi''(x) &= x^2\left[x^{r-2}\sum_{k=0}^{\infty}c_k(k+r)(k+r-1)x^k\right]\\
&= x^r\left[\sum_{k=0}^{\infty}c_k(k+r)(k+r-1)x^k\right].
\end{aligned}$$

Thus

$$\begin{aligned}
L(\phi)(x) &= x^2\phi'' + a(x)x\phi' + b(x)\phi\\
&= x^r\left[\sum_{k=0}^{\infty}c_k(k+r)(k+r-1)x^k\right] + x^r\sum_{k=0}^{\infty}\tilde{\alpha}_k(x)^k + x^r\sum_{k=0}^{\infty}\tilde{\beta}_k x^k\\
&= x^r\sum_{k=0}^{\infty}\left[(k+r)(k+r-1)c_k + \tilde{\alpha}_k + \tilde{\beta}_k\right]x^k,
\end{aligned}$$

and we must have

$$[\quad]_k = \left[(k+r)(k+r-1)c_k + \tilde{\alpha}_k + \tilde{\beta}_k\right] = 0, \qquad (k = 0, 1, 2, \cdots).$$

Using the definitions of $\tilde{\alpha}_k, \tilde{\beta}_k$ we can write the bracket $[\quad]_k$ as

$$\begin{aligned}
[\quad]_k &= (k+r)(k+r-1)c_k + \sum_{j=0}^{k}c_j(j+r)\alpha_{k-j} + \sum_{j=0}^{k}c_j\beta_{k-j}\\
&= (k+r)(k+r-1)c_k + c_k(k+r)\alpha_0 + c_k\beta_0 + \sum_{j=0}^{k-1}\left[(j+r)\alpha_{k-j} + \beta_{k-j}\right]c_j\\
&= (k+r)(k+r-1)c_k + \tilde{\alpha}_k + \tilde{\beta}_k\\
&= (k+r)(k+r-1)c_k + \sum_{j=0}^{k}c_j + \sum_{j=0}^{k}c_j(j+r)\alpha_{k-j} + \sum_{j=0}^{k}c_j\beta_{k-j}\\
&= (k+r)(k+r-1)c_k + c_k(k+r)\alpha_0 + c_k\beta_0 + \sum_{j=0}^{k-1}\left[(j+r)\alpha_{k-j} + \beta_{k-j}\right]c_j.
\end{aligned}$$

For $k = 0$ we must have

$$r(r-1) + r\alpha_0 + \beta_0 = 0, \tag{4.32}$$

since $c_0 \neq 0$. The second degree polynomial q given by

$$q(r) = r(r-1) + r\alpha_0 + \beta_0$$

is called the indicial polynomial for 4.30, and the only admissible values of r are the roots of q. We see that

$$[\quad]_k = q(r+k)c_k + d_k = 0, \quad (k = 1, 2, \cdots), \tag{4.33}$$

where

$$d_k = \sum_{j=0}^{k-1} \left[(j+r)\alpha_{k-j} + \beta_{k-j}\right] c_j, \quad (k = 1, 2, \cdots). \tag{4.34}$$

Note that d_k is a linear combination of $c_0, c_1, \cdots, c_{k-1}$ with coefficients involving the known functions a, b and r. Leaving r and c_0 indeterminate for the moment we solve the equations 4.33, 4.34 successively in terms of c_0 and r. The solutions we denote by $C_k(r)$, and the corresponding d_k by $D_k(r)$. Put $k = 1$ in the equation 4.34, we get

$$\begin{aligned} D_1(r) = d_1 &= \sum_{j=0}^{0} \left[(j+r)\alpha_{1-j} + \beta_{1-j}\right] c_j \\ &= (0+r)\alpha_{1-0} + \beta_{1-0}c_0 \\ D_1(r) &= (r\alpha_1 + \beta_1)c_0, \end{aligned} \tag{4.35}$$

$$\begin{aligned} q(r+1)C_1(r) + D_1(r) &= 0, \\ C_1(r) &= -\frac{D_1(r)}{q(r+1)}. \end{aligned} \tag{4.36}$$

Put $k = 2$, we get

$$D_2(r) = d_2 = \sum_{j=0}^{1} \left[(j+r)\alpha_{2-j} + \beta_{2-j}\right] c_j \tag{4.37}$$

$$\begin{aligned} q(r+2)C_2(r) + D_2(r) &= 0, \\ C_2(r) &= -\frac{D_2(r)}{q(r+2)}, \end{aligned} \tag{4.38}$$

and in general

$$D_k(r) = \sum_{j=0}^{k-1} \left[(j+r)\alpha_{k-j} + \beta_{k-j}\right] c_j, \tag{4.39}$$

$$C_k(r) = -\frac{D_k(r)}{q(r+k)}, \quad (k = 1, 2, \cdots). \tag{4.40}$$

The C_k thus determined are rational functions of r (quotients of polynomials), and the only points where they cease to exist are the points r for which $q(r+k)=0$ for some $k=1,2,\cdots$. Only two such possible points exist. Let us define ϕ by

$$\Phi(x,r)=c_0x^r+x^r\sum_{k=1}^{\infty}C_k(r)x^k. \tag{4.41}$$

If the series in 4.41 converges for $0<x<r_0$, then clearly

$$L(\Phi)(x,r)=c_0q(r)x^r. \tag{4.42}$$

If the ϕ given by 4.29 is a solution of 4.30 then r must be a root of the indicial polynomial q, and the $c_k(k\geq 1)$ are determined uniquely in terms of r and c_0 to be the $C_k(r)$ of 4.40, provided $q(r+k)\neq 0$ for $k=1,2,\cdots$.

Conversely if r is a root of q, and if the $C_k(r)$ can be determined, then the function ϕ given by $\phi(x)=\Phi(x,r)$ is a solution of 4.30 for any choice of c_0, provided the series in 4.41 can be shown to be convergent.

Let r_1,r_2 be the two roots of q, and suppose we have labeled them so that $Re\ r_1\geqq Re\ r_2$. Then $q(r_1+k)\neq 0$ for any $k=1,2,\cdots$.

Thus $C_k(r_1)$ exists for all $k=1,2,\cdots$, and letting $c_0=C_0(r_1)=1$ we see that the function ϕ_1 given by

$$\phi_1(x)=x^{r_1}\sum_{k=0}^{\infty}C_k(r_1)x^k \quad (C_0(r_1)=1), \tag{4.43}$$

is a solution of 4.30, provided the series is convergent.

If r_2 is a root of q distinct from r_1, and $q(r_2+k)\neq 0$ for $k=1,2,\cdots$, then clearly $C_k(r_2)$ is defined for $k=1,2,\cdots$, and the function ϕ_2 given by

$$\phi_2(x)=x^{r_2}\sum_{k=0}^{\infty}C_k(r_2)x^k \quad (C_0(r_2)=1), \tag{4.44}$$

is another solution of 4.30, provided the series is convergent. The condition

$$q(r_2+k)\neq 0 \quad \text{for} \quad k=1,2,\cdots$$

is the same as

$$r_1\neq r_2+k \quad \text{for} \quad k=1,2,\cdots,$$

or r_1-r_2 is not a positive integer.

As we have seen in 4.43, 4.44, the coefficients $c_k,\tilde{c}_k$ appearing in the solutions ϕ_1,ϕ_2 of the above theorem are given by

$$c_k=C_k(r_1),\quad \tilde{c}_k=C_k(r_2)\quad \text{for}\quad k=1,2,\cdots,$$

where the $C_k(r)$, $(k=1,2,\cdots)$, are the solutions of the equations 4.39, 4.40, with $C_0(r)=1$.

In the case of the Euler equation, that the calculations made for $x>0$ remain valid for $x<0$ provided x^r is replaced every where by $|x|^r$.

If r_1-r_2 is either zero or a positive integer we shall say that we have an exceptional case. The Euler equation shows that if $r_ß=r_2$ we must expect solutions involving $\log x$. In the case when r_1-r_2 is a positive integer $\log x$ may appear.

4.3.2 The exceptional cases

Theorem 4.4 *Consider the equation*

$$x^2y'' + a(x)xy' + b(x)y = 0,$$

where a and b have power series expansions which are convergent for $|x| < r_2$, $r_0 > 0$. Let r_1, r_2 $(Re\ r_1 \geq Re\ r_2$ be the roots of the indicial polynomial

$$q(r) = r(r-1) + a(0)r + b(0).$$

If $r_1 = r_2$, there are two linearly independent solutions ϕ_1, ϕ_2 for $0 < |x| < r_0$ of the form

$$\phi_1(x) = |x|^{r_1}\sigma_1(x), \quad \phi_2(x) = |x|^{r_1+1}\sigma_2(x) + (\log|x|)\phi_1(x),$$

where σ_1, σ_2 have power series expansions which are convergent for $|x| < r_0$ and $\sigma_1(0) \neq 0$.
If $r_1 - r_2$ is a positive integer, then there are two linearly independent solutions ϕ_1, and ϕ_2 for $0 < |x| < r_0$ of the form

$$\phi_1(x) = |x|^{r_1}\sigma_1(x), \quad \phi_2(x) = |x|^{r_2}\sigma_2(x) + c(\log|x|)\phi_1(x),$$

where σ_1, σ_2 have power series expansions which are convergent for

$$|x| < r_0, \quad \sigma_1(0) \neq 0, \quad \sigma_2(0) \neq 0,$$

and c is a constant. It may happen that $c = 0$.

Proof:
We divide the exceptional cases into two groups according as the root $r_1, r_2(Re\ r_1 \geq Re r_1)$ of the indicial polynomial satisfy

$$(i)\ r_1 = r_2$$
$$(ii)\ r_1 - r_2 \text{ is a positive integer.}$$

We try to find solutions for $0 < x < r_0$. We are going to work in a purely formal way in order to discover the form that the solutions should take. For such x we have from 4.41, 4.42

$$L(\Phi)(x,r) = c_0 q(r) x^r, \tag{4.45}$$

where Φ is given by

$$\Phi(x,r) = c_0 x^r + x^r \sum_{k=1}^{\infty} C_k(r) x^k. \tag{4.46}$$

The $C_k(r)$ are determined recursively by the formulas

$$\begin{aligned} C_0(r) &= c_0 \neq 0, \\ q(r+k)C_k(r) &= -D_k(r), \end{aligned} \tag{4.47}$$

$$D_k(r) = \sum_{j=0}^{k-1} [(j+r)\alpha_{k-j} + \beta_{k-j}]\, C_j(r), \quad (k = 1, 2, \cdots);$$

see 4.39, 4.40.

In case (i) we have

$$q(r_1) = 0, \quad q'(r_1) = 0,$$

and this suggest formally differentiating 4.45 with respect to r. We obtain

$$\begin{aligned} \frac{\partial}{\partial r} L(\Phi)(x, r) &= L\left(\frac{\partial \Phi}{\partial r}\right)(x, r) \\ &= c_0[q'(r) + (\log x) q(r)] x^r, \end{aligned}$$

and we see that if $r = r_1 = r_2, c_0 = 1$, then

$$\phi_2(x) = \frac{\partial \Phi}{\partial r}(x, r_1)$$

will yield a solution of our equation, provided the series involved converge. Computing formally from 4.46 we find

$$\begin{aligned} \phi_2(x) &= x^{r_1} \sum_{k=0}^{\infty} C_k'(r_1) x^k + (\log x) x^{r_1} \sum_{k=0}^{\infty} C_k(r_1) x^k \\ &= x^{r_1} \sum_{k=0}^{\infty} C_k'(r_1) x^k + (\log x) \phi_1(x), \end{aligned}$$

where ϕ_1 is the solution already obtained:

$$\phi_1(x) = x^{r_1} \sum_{k=0}^{\infty} C_k(r_1) x^k, \quad (C_0(r_1) = 1).$$

Note that $C_k'(r_1)$ exists for all $(k = 0, 1, 2, \cdots)$, since C_k is a rational function of r whose denominator is not zero at $r = r_1$. Also $C_0(r) = 1$ implies that $C_0'(r_1) = 0$, and thus the series multiplying x^{r_1} in ϕ_2 starts with the first power of x.
Let us now turn to the case (ii), and suppose that $r_1 = r_2 + m$, where m is a positive integer. If c_0 is given,

$$C_1(r_2), \cdots, C_{m-1}(r_2)$$

all exist as finite numbers, but since

$$q(r+m) C_m(r) = -D_m(r), \tag{4.48}$$

we run into trouble in trying to compute $C_m(r_2)$. Now

$$q(r) = (r - r_1)(r - r_2),$$

and hence

$$q(r+m) = (r - r_2)(r + m - r_2).$$

If $D_m(r)$ also has $r - r_2$ as a factor (i.e.,$D_m(r_2) = 0$) this would cancel the same factor in $q(r+m)$, and 4.48 would give $C_m(r_2)$ as a finite number. Then

$$C_{m+1}(r_2),\ C_{m+2}(r_2), \cdots$$

all exist. In this rather special situation we will have a solution ϕ_2 of the form

$$\phi_2(x) = x^{r_2} \sum_{k=0}^{\infty} C_k(r_2) x^k, \quad (C_0(r_2) = 1).$$

We can always arrange it so that $D_m(r_2) = 0$ by choosing

$$C_0(r) = r - r_2$$

From 4.47, we see that $D_k(r)$ is linear homogeneous in

$$C_0(r), \cdots, C_{k-1}(r),$$

and hence $D_k(r)$ has $c_0(r) = r - r_2$ as a factor. Thus $C_m(r_2)$ will exist as a finite number. Letting

$$\Psi(x,r) = x^r \sum_{k=0}^{\infty} C_k(r) x^k, \quad (C_0(r) = r - r_2), \tag{4.49}$$

we find formally that

$$L(\Psi)(x,r) = (r - r_2) q(r) x^r. \tag{4.50}$$

Putting $r = r_2$ we obtain formally a solution ψ given by

$$\psi(x) = \Psi(x, r_2).$$

However $C_0(r_2) = C_1(r_2) = \cdots = C_{m-1}(r_2) = 0$. Thus the series for ψ actually starts with the m-th power of x, and hence ψ has the form

$$\psi(x) = x^{r_2+m}\sigma(x) = x^{r_1}\sigma(x),$$

where σ is some power series. It is not difficult to see that ψ is just a constant multiple of the solution ϕ_1 already obtained.

To get a solution really associated with r_2 we differentiate 4.47 with respect to r, obtaining

$$\begin{aligned}\frac{\partial}{\partial r} L(\Psi)(x,r) &= L\left(\frac{\partial \Psi}{\partial r}\right)(x,r) \\ &= q(r)x^r + (r - r_2)\left[q'(r) + (\log x) q(r)\right] x^r.\end{aligned}$$

Now letting $r = r_2$ we find that the ϕ_2 given by

$$\phi_2(x) = \frac{\partial \Psi}{\partial r}(x, r_2)$$

is a solution, provided the series involved are convergent. It has the form

$$\phi_2(x) = x^{r_2} \sum_{k=0}^{\infty} C'_k(r_2) x^k + (\log x) x^{r_2} \sum_{k=0}^{\infty} C_k(r_2) x^k,$$

where $C_0(r) = r - r_2$. Since

$$C_0(r_2) = \cdots = C_{m-1}(r_2) = 0,$$

we may write this as

$$\phi_2(x) = x^{r_2} \sum_{k=0}^{\infty} C'_k(r_2) x^k + c(\log x)\phi_1(x),$$

where $c = C_m(r_2)$.
The method used in this section to obtain solutions is called the Frobenius method. All the series obtained converge for $|x| < r_0$, and the ϕ_2 computed formally will be a solution in both the cases (i) and (ii). This requires justifying the differentiating of the various series term by term with respect to r, and this can be done.

Solutions for $x < 0$ can be obtained by replacing

$$x^{r_1}, \quad x^{r_2}, \quad \log x$$

everywhere by

$$|x|^{r_1}, \quad |x|^{r_2}, \quad \log |x|$$

respectively.

Let us sum up

1. We have defined the significance of the Frobenius method.
2. We have discussed the method of finding exceptional regular singular points using Frobenius method.
3. We have discussed the linear homogeneous equation with a regular singular point at the origin always having a solution.
4. We have discussed the general case for a second-order equation with regular singular points.
5. Finally, we figure out some illustrative examples.

Check your progress

3. For the differential equation $x^2y'' - 5y' + 3x^2y = 0$,
   (a) $x = 1$, regular singular point (b) $x = 0$, not regular singular point
   (c) $x = 1$, not regular singular point (d) $x = 0$, regular singular point

4. For the differential equation $x^2y'' + (sinx)y' + (cosx)y = 0$, which of the following statement is true?
   (a) $x = 0$, regular (b) $x = 1$, regular
   (c) $x = 0$, irregular (d) $x = 1$, irregular

4.4 The Bessel equation

In this section, you will learn about the standard forms and notations of the Bessel's equation and functions. The Bessel differential equation is the linear second order ordinary differential equation. The solutions to the Bessel differential equation define the Bessel's functions $J_n(x)$ and $Y_n(x)$ which has a regular singularity at 0 (zero) and an irregular singularity at ∞.

If α is a constant, $Re\ \alpha \geqq 0$, the Bessel equation of order α is the equation

$$L(y) = x^2y'' + xy' + (x^2 - \alpha^2)y = 0. \tag{4.51}$$

This has the form

$$x^2y'' + a(x)y' + b(x)y = 0, \tag{4.52}$$

with $a(x) = x, b(x) = x^2 - \alpha^2$. Since a, b are analytic at $x = 0$, the Bessel equation has the origin as a regular singular point. The indicial polynomial q is given by

$$q(r) = r(r-1) + r - \alpha^2 = r^2 - \alpha^2,$$

whose 2 roots r_1, r_2 are

$$r_1 = \alpha, r_2 = -\alpha.$$

4.4.1 First kind of Bessel equation of order zero:

Let us consider the case $\alpha = 0$. Since the roots are both equal to zero in this case there are two solutions ϕ_1, ϕ_2 of the form

$$\phi_1(x) = \sigma_1(x),$$
$$\phi_2(x) = x\sigma_2(x) + (\log x)\phi_1(x),$$

where σ_1, σ_2 have power series expansion, which converge for all finite x. Let us compute σ_1, σ_2. Let

$$L(y) = x^2y'' + xy' + x^2y,$$

and suppose

$$\sigma_1(x) = \sum_{k=0}^{\infty} c_k x^k, \qquad (c_0 \neq 0).$$

We find

$$\sigma_1'(x) = \phi_1'(x) = \sum_{k=1}^{\infty} c_k k x^{k-1},$$
$$\sigma_1''(x) = \phi_1''(x) = \sum_{k=2}^{\infty} c_k k(k-1) x^{k-2},$$
$$x\sigma_1'(x) = \sum_{k=1}^{\infty} c_k k x^k = c_1 x + \sum_{k=2}^{\infty} c_k k x^k,$$

$$x^2\sigma_1''(x) = \sum_{k=2}^{\infty} c_k k(k-1)x^k,$$
$$x^2\sigma_1(x) = \sum_{k=0}^{\infty} c_k x^{k+2} = \sum_{k=2}^{\infty} c_{k-2}x^k.$$

We obtain

$$\phi_1'(x) = \sum_{k=1}^{\infty} kc_k x^{k-1} + \frac{\phi_1(x)}{x} + (\log x)\phi_1'(x),$$
$$\phi_2''(x) = \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - \frac{\phi_1(x)}{x^2} + \frac{2}{x}\phi_1'(x) + (\log x)\phi_1''(x).$$

Thus

$$L(\sigma_1)(x) = c_1(x) + \sum_{k=2}^{\infty}\{[k(k-1)+k]c_k + c_{k-2}\}x^k = 0.$$

We see that

$$c_1 = 0$$
$$[k(k-1)+k]c_k + c_{k-2} = 0, \quad (k = 2, 3, \cdots).$$

The second set of equations is the same as

$$c_k = -\frac{c_{k-2}}{k^2}, \quad (k = 2, 3, \cdots).$$

The choice $c_0 = 1$ implies

$$c_2 = -\frac{1}{2^2}, \;\; c_4 = -\frac{c_2}{4^2} = \frac{1}{2^2 \cdot 4^2}, \cdots,$$

and in general

$$c_{2m} = \frac{(-1)^m}{2^2 \cdot 4^2 \cdots (2m)^2} = \frac{(-1)^m}{2^{2m}(m!)^2}, \quad (m = 1, 2, \cdots).$$

Since $c_1 = 0$ we have

$$c_3 = c_5 = \cdots = 0.$$

Thus σ_1 contains only even powers of x, and we obtain

$$\sigma_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2},$$

where as usual $0! = 1$, and $2^0 = 1$. The function defined by this series is called the Bessel function of zero order of the first kind and is denoted by J_0. Thus

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2}\left(\frac{x}{2}\right)^{2m}.$$

It is easily checked by the ratio test that this series indeed converges for all finite x.

4.4.2 Second kind of Bessel equation of order zero:

We now determine a second solution ϕ_2 for the Bessel equation of order zero. Letting $\phi_1 = J_0$ this solution has the form,

$$\phi_2(x) = \sum_{k=0}^{\infty} c_k x^k + (\log x)\phi_1(x), \quad (c_0 = 0).$$

We obtain

$$\begin{aligned}
\phi_2'(x) &= \sum_{k=1}^{\infty} c_k k x^{k-1} + \frac{1}{x}\phi_1(x) + \phi_1'(x)(\log x), \\
\phi_2''(x) &= \sum_{k=2}^{\infty} c_k k(k-1)x^{k-2} - \phi_1(x)\left(\frac{-1}{x^2}\right) + \frac{2}{x}\phi_1'(x) + \phi_1''(x)(\log x).
\end{aligned}$$

Thus

$$\begin{aligned}
L(\phi_2)(x) &= x^2\phi_2''(x) + x\phi_2'(x) + x^2\phi_2(x) \\
&= c_1 x + 2^2 c_2 x^2 + \sum_{k=3}^{\infty}\left(k^2 c_k + c_{k-2}\right)x^k + 2x\phi_1'(x) + (\log x)L(\phi_1)(x),
\end{aligned}$$

and since $L(\phi_1)(x) = 0$, we have

$$\begin{aligned}
c_1 x + 2^2 c_2 x^2 + \sum_{k=3}^{\infty}\left(k^2 c_k + c_{k-2}\right)x^k &= -2x\phi_1'(x) \\
&= -2x\sum_{m=1}^{\infty}\frac{(-1)^m 2m x^{2m-1}}{2^{2m}(m!)^2} \\
&= -2\sum_{m=1}^{\infty}\frac{(-1)^m 2m x^{2m}}{2^{2m}(m!)^2}.
\end{aligned}$$

Hence equating the coefficients of x and x^2, we get $c_1 = 0$,

$$\begin{aligned}
m = 1 &\Rightarrow 2^2 c_2 = \frac{-2(-1)2}{2^2(1!)^2} = \frac{4}{4} = 1 \\
m = 2 &\Rightarrow 3^2 c_3 + c_1 = 0 \\
&\quad 3^2 c_3 + 0 = 0 \\
&\quad 3^2 c_3 = 0 \\
&\quad \Rightarrow c_3 = 0,
\end{aligned}$$

and we see that since the series on the right has only even powers of x,

$$c_1 = c_5 = c_7 = \cdots = 0.$$

The recursion relation for the other coefficient is

$$(2m)^2 c_{2m} + c_{2m-2} = \frac{(-1)^{m+1} m}{2^{2m-2}(m!)^2}, \quad (m = 2, 3, \cdots).$$

We have

$$
\begin{aligned}
m = 1 \Rightarrow 2^2 c_2 + c_0 &= \frac{(-1)^2}{2^0 (1!)^2} \\
2^2 c_2 + 0 &= \frac{1}{1} \\
c_2 &= \frac{1}{2^2}. \\
m = 2 \Rightarrow 2^2 2^2 c_4 + c_2 &= -\frac{(1)^3 \cdot 2}{2^2 (2!)^2}
\end{aligned}
$$

$$
\begin{aligned}
4^2 c_4 + \frac{1}{2^2} &= -\frac{(1)}{2 \cdot 2^2} \\
4^2 c_4 &= \frac{1}{2^2} - \frac{(1)}{2 \cdot 2^2} \\
c_4 &= \frac{1}{4^2} \frac{1}{2^2} - \frac{(1)}{2 \cdot 2^2} \\
c_4 &= -\frac{1}{4^2 \cdot 2^2} \left[1 + \frac{1}{2} \right]
\end{aligned}
$$

and

$$
\begin{aligned}
c_6 &= \frac{1}{6^2} \left[\frac{1}{4^2 \cdot 2^2} \left(1 + \frac{1}{2} \right) + \frac{1}{4^2 \cdot 2^2} \left(1 + \frac{1}{3} \right) \right] \\
&= \frac{1}{2^2 \cdot 4^2 \cdot 6^2} \left[1 + \frac{1}{2} + \frac{1}{3} \right] \\
&= \frac{1}{2^2 (1^2 \cdot 2^2 \cdot 3^2)} \left[1 + \frac{1}{2} + \frac{1}{3} \right].
\end{aligned}
$$

It can be shown by induction that

$$
c_{2m} = \frac{(-1)^{m-1}}{2^{2m} (m!)^2} \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} \right], \quad (m = 1, 2, \cdots).
$$

Then

$$
\begin{aligned}
\phi_2(x) &= \sum_{k=0}^{\infty} c_k x^k + (\log x) \phi_1(x), \\
\phi_2(x) &= \sum_{m=0}^{\infty} c_{2m} x^{2m} + (\log x) \phi_1(x), \\
\phi_2(x) &= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(m!)^2} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} \right) \left(\frac{x}{2} \right)^{2m} + (\log x) \phi_1(x).
\end{aligned}
$$

The solution thus determined is called a Bessel function of zero order of the second kind, and is denoted by K_0. Hence

$$
K_0(x) = -\sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} \right) \left(\frac{x}{2} \right)^{2m} + (\log x) J_0(x).
$$

4.4.3 Bessel function of order α:

Now we compute solutions for the Bessel equation of order α, where $\alpha \neq 0$, and $Re\ \alpha \geq 0$:

$$L(y) = x^2 y'' + xy' + (x^2 - \alpha^2)y = 0. \tag{4.53}$$

This is of the form

$$x^2 y'' + a(x)y' + b(x)y = 0. \tag{4.54}$$

As before we restrict attention to the case $x > 0$. The indicial polynomial is given by

$$\begin{aligned} q(r) = r(r-1) + r + \alpha^2 &= 0 \\ r^2 - r + r - \alpha^2 &= 0 \\ r &= \pm\alpha. \end{aligned}$$

The roots of the indicial polynomial equation are

$$r_1 = \alpha, \;\; r_2 = -\alpha.$$

First we determine a solution corresponding to the root $r_1 = \alpha$. The solution ϕ_1 has the form

$$\begin{aligned} \phi_1(x) &= x^{r_1} \sum_{k=0}^{\infty} c_k x^k, \\ &= x^{\alpha} \sum_{k=0}^{\infty} c_k x^k, \\ \phi_1(x) &= \sum_{k=0}^{\infty} c_k x^{k+\alpha}, \qquad (c_0 \neq 0). \end{aligned}$$

Let $y = \phi_1(x)$. Then

$$L(\phi_1)(x) = x^2 \phi_1''(x) + x\phi_1'(x) + (x^2 - \alpha^2)\phi_1(x) = 0,$$

and

$$\begin{aligned} \phi_1(x) &= \sum_{k=0}^{\infty} c_k x^{k+\alpha} \\ &= c_0 x^{\alpha} + c_1 x^{\alpha+1} + \sum_{k=2}^{\infty} c_k x^{k+\alpha} \\ \phi_1'(x) &= \sum_{k=0}^{\infty} c_k (k+\alpha) x^{k+\alpha-1} \\ &= c_0(\alpha)x^{\alpha-1} + c_1(1+\alpha)x^{\alpha} + \sum_{k=2}^{\infty} c_k(k+\alpha)x^{k+\alpha-1} \end{aligned}$$

$$
\begin{aligned}
x\phi_1'(x) &= c_0(\alpha)x^\alpha + c_1(1+\alpha)x^{\alpha+1} + \sum_{k=2}^{\infty} c_k(k+\alpha)x^{k+\alpha} \\
\phi_1''(x) &= \sum_{k=0}^{\infty} c_k(k+\alpha)(k+\alpha-1)x^{k+\alpha-2} \\
&= c_0(\alpha)(\alpha-1)x^{\alpha-2} + c_1(\alpha+1)(\alpha)x^{\alpha-1} + \sum_{k=2}^{\infty} c_k(k+\alpha)(k+\alpha-1)x^{k+\alpha-2} \\
x^2\phi_1''(x) &= c_0(\alpha)(\alpha-1)x^{\alpha} + c_1(\alpha+1)(\alpha)x^{\alpha+1} + \sum_{k=2}^{\infty} c_k(k+\alpha)(k+\alpha-1)x^{k+\alpha} \\
(x^2-\alpha^2)\phi_1(x) &= x^2\left[\sum_{k=2}^{\infty} c_{k-2}x^{k+\alpha}\right] - \alpha^2\left[c_0x^\alpha + c_1x^{\alpha+1} + \sum_{k=2}^{\infty} c_kx^{k+\alpha}\right] \\
&= \sum_{k=2}^{\infty} c_{k-2}x^{k+\alpha+2} - \alpha^2c_0x^\alpha - \alpha^2c_1x^{\alpha+1} - \alpha^2\sum_{k=2}^{\infty} c_kx^{k+\alpha}.
\end{aligned}
$$

We find that

$$
L(\phi_1)(x) = 0\cdot c_0x^\alpha + \left[(\alpha+1)^2-\alpha^2\right]c_1x^{\alpha+1} + x^\alpha\sum_{k=2}^{\infty}\{\left[(\alpha+k)^2-\alpha^2)\right]c_k + c_{k-2}\}x^k = 0.
$$

Thus we have

$$
\begin{aligned}
c_1x^{\alpha+1}\left[2\alpha+1\right] &= 0 \\
\Rightarrow c_1 &= 0. \qquad (4.55) \\
c_k\left[k^2+2k\alpha+c_{k-2}\right] &= 0 \\
c_k\left[k^2+2k\alpha\right] &= -c_{k-2} \\
c_k &= -\frac{c_{k-2}}{k^2+2k\alpha}, \qquad (k=2,3,\cdots). \qquad (4.56)
\end{aligned}
$$

We find

$$
\begin{aligned}
k=2, \quad c_2 &= -\frac{c_0}{2^2+2^2\alpha} = -\frac{c_0}{2^2(1+\alpha)}, \\
k=3, \quad c_3 &= -\frac{c_1}{3^2+3(2)\alpha} = 0, \\
k=4, \quad c_4 &= -\frac{c_2}{4^2+2(4)\alpha} \\
&= \frac{c_0}{8(2+\alpha).2^2(1+\alpha)} \\
&= \frac{c_0}{2^4.(2!)(\alpha+2)(\alpha+1)},
\end{aligned}
$$

$$
\begin{aligned}
k=5, \quad c_5 &= -\frac{c_1}{5^2+5(2)\alpha} = 0, \\
k=6, \quad c_6 &= -\frac{c_4}{6^2+2(6)\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{c_0}{6(6+2\alpha).2^4.(2!)(\alpha+2)(\alpha+1)}, \\
&= -\frac{c_0}{6(2).2^4.(2!)(\alpha+3)(\alpha+2)(\alpha+1)}, \\
&= -\frac{c_0}{2^6.3!(\alpha+3)(\alpha+2)(\alpha+1)}.
\end{aligned}
$$

This implies that

$$c_1 = c_3 = c_5 = \cdots = 0.$$

In general,

$$c_{2m} = \frac{(-1)^m c_0}{2^{2m}(m!)(\alpha+1)(\alpha+2)(\alpha+3)\cdots(\alpha+m)}.$$

Our solution thus becomes

$$\phi_1(x) = c_0 x^\alpha + c_0 x^\alpha \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)(\alpha+1)(\alpha+2)(\alpha+3)\cdots(\alpha+m)}. \tag{4.57}$$

For $\alpha = 0, c_0 = 1$, this reduces to $J_0(x)$. It is usual to choose

$$c_0 = \frac{1}{2^\alpha \Gamma(\alpha+1)}, \tag{4.58}$$

where Γ is the gamma function defined by

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx, \quad (Re\, z\, > 0).$$

It is readily seen that

$$\Gamma(z+1) = z\Gamma(z). \tag{4.59}$$

Indeed, integrating by parts, we have:

$$
\begin{aligned}
\Gamma(z+1) &= \lim_{T\to\infty} \int_0^T e^{-x} x^z \, dx \\
&= \lim_{T\to\infty} \left[-x^z e^{-x} \Big|_0^T + z \int_0^T e^{-x} x^{z-1} \, dx \right] \\
&= z \lim_{T\to\infty} \int_0^T e^{-x} x^{z-1} \, dx \\
&= z\Gamma(z),
\end{aligned}
$$

since $T^z e^{-T} \to 0$ as $T \to \infty$. Also, since

$$\Gamma(1) = \int_0^\infty e^{-x} \, dx = 1,$$

if z is a positive integer n,

$$\Gamma(n+1) = n!.$$

Thus the gamma function is an extension of the factorial function to numbers which are not integers.

The relation 4.59 can be used to define $\Gamma(z)$ for z such that $Re\ z < 0$, provided z is not a negative integer. To see this suppose N is the positive integer such that

$$-N < Re\ z \leq -N+1.$$

Then $Re\ (z+N) > 0$, and we can define $\Gamma(z)$ in terms of $\Gamma(z+N)$ by

$$\Gamma(z) = \frac{\Gamma(z+N)}{z(z+1)\cdots(z+N-1)}, \quad (Re\ z < 0),$$

provided $z \neq -N+1$. The gamma function is not defined at $0, -1, -2, \cdots$.

Returning to 4.57, if we use the c_0 given by 4.58 we obtain a solution of the Bessel equation of order α which is denoted by J_α, and is called the Bessel function of order α of the first kind:

$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)\Gamma(\alpha+m+1)} \left(\frac{x}{2}\right)^{2m}, \quad (Re\ \alpha \geq 0). \tag{4.60}$$

Notice that this formula for J_α reduces to J_0 when $\alpha = 0$, since $\Gamma(m+1) = m!$.

There are now two cases according as $r_1 - r_2 = \alpha - (-\alpha) = 2\alpha$ is a positive integer or not. If 2α is not a positive integer, there is another solution ϕ_2 of the form

$$\phi_2(x) = x^{-\alpha} \sum_{k=0}^{\infty} c_k x^k.$$

We find that our calculations for the root $r_1 = \alpha$ carry over provided only that we replace α by $-\alpha$ everywhere. Thus

$$J_{-\alpha}(x) = \left(\frac{x}{2}\right)^{-\alpha} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)\Gamma(m-\alpha+1)} \left(\frac{x}{2}\right)^{2m}$$

gives a second solution in case 2α is not a positive integer.

Since $\Gamma(m-\alpha+1)$ exists for $m = 0, 1, 2$ provided α is not a positive integer, we see that $J_{-\alpha}$ exists in this case, even if $r_1 - r_2 = 2\alpha$ is a positive integer. Thus, if α is not zero or a positive integer, J_α and $J_{-\alpha}$ form a basis for the solutions of the Bessel equation of order α for $x > 0$.

The only remaining case is that for which α is a positive integer, say $\alpha = n$. There is a solution ϕ_2 of the form

$$\phi_2(x) = x^{-n} \sum_{k=0}^{\infty} c_k x^k + c(\log x) J_n(x).$$

Now, we find that

$$L(\phi_2)(x) = x^2\phi_2''(x) + x\phi_2'(x) + (x^2 - n^2)\phi_2(x) = 0.$$

First we determine a solution corresponding to the root $\alpha = n$. From the solution ϕ_2 has the form

$$\begin{aligned}\phi_2(x) &= x^{-\alpha}\sum_{k=0}^{\infty} c_k x^k,\\ &= x^{-n}\sum_{k=0}^{\infty} c_k x^k,\\ \phi_2(x) &= \sum_{k=0}^{\infty} c_k x^{k-n}, \qquad (c_0 \neq 0).\end{aligned}$$

Then

$$\begin{aligned}\phi_2(x) &= \sum_{k=0}^{\infty} c_k x^{k-n}\\ &= c_0 x^{-n} + c_1 x^{1-n} + \sum_{k=2}^{\infty} c_k x^{k-n}\\ \phi_2'(x) &= \sum_{k=0}^{\infty} c_k (k-n) x^{k-n-1}\\ &= c_0(-n)x^{-n-1} + c_1(1-n)x^{-n} + \sum_{k=2}^{\infty} c_k(k-n)x^{k-n-1}\\ x\phi_2'(x) &= c_0(-n)x^{-n} + c_1(1-n)x^{1-n} + \sum_{k=2}^{\infty} c_k(k-n)x^{k-n}\end{aligned}$$

$$\begin{aligned}\phi_2''(x) &= \sum_{k=0}^{\infty} c_k(k-n)(k-n-1)x^{k-n-2}\\ &= c_0(-n)(-n-1)x^{-n-2} + c_1(1-n)(-n)x^{-n-1} + \sum_{k=2}^{\infty} c_k(k-n)\\ &\quad (k-n-1)x^{k-n-2}\\ x^2\phi_2''(x) &= c_0(-n)(-n-1)x^{-n} + c_1(1-n)(-n)x^{1-n} + \sum_{k=2}^{\infty} c_k(k-n)(k-n-1)x^{k-n}\\ (x^2-n^2)\phi_2(x) &= x^2\left[\sum_{k=2}^{\infty} c_{k-2}x^{k-n}\right] - n^2\left[c_0 x^{-n} + c_1 x^{1-n} + \sum_{k=2}^{\infty} c_k x^{k-n}\right]\\ &= \sum_{k=2}^{\infty} c_{k-2}x^{k-n+2} - n^2 c_0 x^{-n} - n^2 c_1 x^{1-n} - n^2\sum_{k=2}^{\infty} c_k x^{k-n}.\end{aligned}$$

Thus

$$L(\phi_2(x)) = c_0(-n)(-n-1)x^{-n} + c_1(1-n)(-n)x^{1-n} + \sum_{k=2}^{\infty} c_k(k-n)(k-n-1)x^{k-n}$$

$$+ c_0(-n)x^{-n} + c_1(1-n)x^{1-n} + \sum_{k=2}^{\infty} c_k(k-n)x^{k-n} + \sum_{k=2}^{\infty} c_{k-2}x^{k-n+2}$$

$$- n^2c_0x^{-n} - n^2c_1x^{1-n} - n^2\sum_{k=2}^{\infty} c_kx^{k-n} + 2cxJ_n'(x) + c(\log x)L(J_n)(x) = 0,$$

$$= c_0x^{-n}\left[-n(-n-1) - n - n^2\right] + c_1x^{1-n}\left[-n(1-n) - n + 1 - n^2\right]$$

$$+ \sum_{k=2}^{\infty}\left[c_k\left[(k-n)(k-n-1) + k - n - n^2\right] + c_{k-2}\right]x^{k-n}$$

$$+ 2cxJ_n'(x) + c(\log x)L(J_n)(x) = 0,$$

$$= c_0x^{-n}\left[-n^2 + n + n^2 - n\right] + c_1x^{1-n}\left[-n^2 - n - n + 1 + n^2\right]$$

$$+ \sum_{k=2}^{\infty}\left[c_k\left[k^2 - kn - k - kn + k - n - n^2 + n + n^2\right] + c_{k-2}\right]$$

$$x^{k-n} + 2cxJ_n'(x) + c(\log x)L(J_n)(x) = 0$$

$$= c_0x^{-n} + [(1-n)^2 - n^2]c_1x^{1-n} + x^{-n}\sum_{k=1}^{\infty}\{[(k-n)^2 - n^2]c_k + c_{k-2}\}x^k$$

$$+ 2cxJ_n'(x) + c(\log(x))L(J_n)(x) = 0,$$

and since $L(J_n)(x) = 0$, we have, on multiplying by x^n,

$$(1-2n)c_1x + \sum_{k=2}^{\infty}[k(k-2n)c_k + c_{k-2}]x^k = -2c\sum_{m=0}^{\infty}(2m+n)d_{2m}x^{2m+2n}. \tag{4.61}$$

Here we have put

$$J_n(x) = \sum_{m=0}^{m} d_{2m}x^{2m+n}, \tag{4.62}$$

and hence

$$d_{2m} = \frac{(-1)^m}{2^{2m+n}m!(m+n)!}. \tag{4.63}$$

The series on the right side of 4.61 begins with x^{2n}, and since n is a positive integer, we have $c_1 = 0$. Further, if $n > 1$,

$$k(k-2n)c_k + c_{k-2} = 0, \quad (k = 2, 3, \ldots, 2n-1).$$

Equating the coefficients of x^{-n}, x^{1-n} and x^{k-n},we get

$$\begin{aligned} c_1x^{1-n}[1-2n] &= 0 \\ \Rightarrow c_1 &= 0. \end{aligned} \tag{4.64}$$

$$\begin{aligned} c_k\left[k^2 - 2kn + c_{k-2}\right] &= 0 \\ c_k\left[k^2 - 2kn\right] &= -c_{k-2} \\ c_k &= -\frac{c_{k-2}}{k^2 - 2kn} \quad (k = 2, 3, \cdots). \end{aligned} \tag{4.65}$$

Then

$$\begin{aligned}
k=2, \quad & c_2 = -\frac{c_0}{2^2 n - 2^2} = \frac{c_0}{2^2(n-1)}, \\
k=3, \quad & c_3 = -\frac{c_1}{3^2 - 3(2)n} = 0, \\
k=4, \quad & c_4 = -\frac{c_2}{2(4)n - 4^2} \\
& = \frac{c_0}{8(n-2)\cdot 2^2(n-1)} \\
& = \frac{c_0}{2^4\cdot(2!)(n-2)(n-1)},
\end{aligned}$$

$$\begin{aligned}
k=5, \quad & c_5 = -\frac{c_1}{5^2 - 5(2)n} = 0, \\
k=6, \quad & c_6 = \frac{c_4}{2(6)n - 6^2} \\
& = \frac{c_0}{6(2n-6)\cdot 2^4\cdot(2!)(n-2)(n-1)}, \\
& = \frac{c_0}{6(2)\cdot 2^4\cdot(2!)(n-3)(n-2)(n-1)}, \\
& = \frac{c_0}{2^6\cdot 3(2!)(n-3)(n-2)(n-1)}.
\end{aligned}$$

Finally, we get

$$c_1 = c_3 = c_5 = \cdots = c_{2n-1} = 0.$$

In general,

$$c_{2j} = \frac{c_0}{2^{2j}(j!)(n-1)(n-2)(n-3)\cdots(n-j)}, \quad (j = 1, 2, \cdots, n-1). \tag{4.66}$$

Comparing the coefficients of x^{2n} in 4.61 we obtain:

$$c_{2n-1} = -2cnd_0 = -\frac{c}{2^{n-1}(n-1)!}.$$

On the other hand from 4.66 it follows that

$$c_{2n-2} = \frac{c_0}{2^{2n-1}(n-1)!(n-1)!},$$

and therefore

$$c = -\frac{c_0}{2^{n-1}(n-1)!}. \tag{4.67}$$

Since the series on the right side of 4.61 contains only even powers of x the same must be true of the series on the left side of 4.61, and this implies

$$c_{2n+1} = c_{2n+3} = \cdots = 0.$$

The coefficient c_{2n} is undetermined, but the remaining coefficients

$$c_{2n+3}, \quad c_{2n+1}, \cdots$$

are obtained from the equations:

$$2m(2n+2m)c_{2n+2m} + c_{2n+2m-3} = -2c(n+2m)d_{2m}, \quad (m = 1, 2, \cdots).$$

For $m = 1$ we have

$$c_{2n+2} = -\frac{cd_1}{2}\left(1 + \frac{1}{n+1}\right) - \frac{c_{2n}}{4(n+1)}.$$

We now choose c_{2n} so that

$$\frac{c_{2n}}{4(n+1)} = \frac{cd_2}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right),$$

since $4(n+1)d_2 = -d_0$,

$$c_{2n} = -\frac{cd_0}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).$$

With the choice of c_{2n}, we have

$$c_{2n+2} = -\frac{cd_2}{2}\left(1 + 1 + \frac{1}{2} + \cdots + \frac{1}{n+1}\right).$$

For $m = 2$ we obtain

$$c_{2n+4} = -\frac{cd_4}{2}\left(\frac{1}{2} + \frac{1}{n+2}\right) - \frac{c_{2n+2}}{2^2 \cdot 2 \cdot (n+2)}.$$

Since $2^2 \cdot 2 \cdot (n+2)d_4 = -d_2$,

$$\frac{c_{2n+2}}{2^2 \cdot 2 \cdot (n+2)} = \frac{cd_4}{2}\left(1 + 1 + \frac{1}{2} + \frac{1}{n+1}\right),$$

and therefore

$$c_{2n+4} = -\frac{cd_4}{2}\left(1 + \frac{1}{2} + 1 + \frac{1}{2} + \cdots + \frac{1}{n+2}\right).$$

It can be shown by induction that

$$c_{2n+2m} = -\frac{cd_{2m}}{2}\left[\left(1 + \frac{1}{2} + \cdots + \frac{1}{m}\right) + \left(1 + \frac{1}{2} + \cdots + \frac{1}{n+m}\right)\right], \quad (m = 1, 2, \cdots).$$

Finally, we obtain for our solution ϕ_2 the function given by

$$\phi_2(x) = c_0 x^{-n} + c_0 x^{-n} \sum_{j=1}^{n-1} \frac{x^{2j}}{2^{2j}(j!)(n-1)(n-2)(n-3)\cdots(n-j)}$$

$$-\frac{cd_0}{2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)x^n$$
$$-\frac{c}{2}\sum_{m=1}^{\infty} d_{2m}\left[\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right)+\left(1+\frac{1}{2}+\cdots+\frac{1}{n+m}\right)\right]x^{n+2m}+c(\log x)J_n(x),$$

where c_0 and c are constants related by 4.67 and d_{2m} is given by 4.63. When $c=1$ the resulting function ϕ_2 is often denoted by K_n. In this case

$$c_0 = -2^{(n-1)}(n-1)!,$$

and therefore we may write

$$K_n(x) = -\frac{1}{2}\left(\frac{x}{2}\right)^{-n}\sum_{j=0}^{n-1}\frac{(n-j-1)}{j!}\left(\frac{x}{2}\right)^{2j} - \frac{1}{2}\frac{1}{n!}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\left(\frac{x}{2}\right)^n -$$
$$\frac{1}{2}\left(\frac{x}{2}\right)^n\sum_{m=1}^{\infty}\frac{(-1)^m}{m!(m+n)!}\left[\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right)+\left(1+\frac{1}{2}+\cdots+\frac{1}{m+n}\right)\right]\left(\frac{x}{2}\right)^{2m}$$
$$+(\log x)J_n(x).$$

This formula reduces to the one for $K_0(x)$ when $n=0$, provided we interpret the first two sums on the right as zero in this case. The function K_n is called a Bessel function of order n of the second kind.

Let us sum up

1. We have discussed the Bessel's equation of order zero and α.
2. We have discussed the gamma function and the relation between the gamma and exponential functions.

Check your progress

5. The value of $x^{1/2}J_{1/2}(x)$ is

(a) $\frac{\sqrt{2}}{\Gamma\left(\frac{1}{2}\right)}\sin x$ (b) $\frac{\sqrt{2}}{\Gamma\left(\frac{1}{2}\right)}\cos x$ (c) $\frac{\sqrt{\frac{1}{2}}}{\Gamma(2)}\sin x$ (d) $\frac{\sqrt{\frac{1}{2}}}{\Gamma(2)}\cos x$

6. Find out the Bessel's Equation

(a) $x^2y''+xy'+\frac{1}{\alpha}(x^2-\alpha^2)y=0$ (b) $y''+xy'+(x^2-\alpha^2)y=0$
(c) $x^2y''+xy'+(x^2-\alpha^2)y=0$ (d) None of these

Summary

- A power series represents a continuous function within its interval of convergence.
- A power series can be differentiated term wise within its interval of convergence.
- The theory of ordinary differential equations for the complex plane are classified into ordinary points, at which the equation's coefficients are analytic functions, and singular points at which some coefficient has a singularity.

- Any singular point which is not regular is called irregular singular point.
- The singularities of second order linear ODEs have been divided into two kinds, regular singularities and irregular singularities.
- Frobenius Method: Solving around singular points.
- When the Frobenius series is used to solve the differential equation then the parameter must be chosen so that when the series is substituted into the differential equation the coefficient of the smallest power of x is zero. This is called the indicial equation.
- An indicial equation, also called a characteristic equation, is a recurrence equation obtained during application of the Frobenius method of solving a second order ordinary differential equation.
- Bessel functions are solutions to Bessel's differential equation, commonly arising in problems with cylindrical or spherical symmetry in physics and engineering.
- Here, for an arbitrary complex number α, the order of the Bessel function. Although α and $-\alpha$ produce the same differential equation for real α, it is conventional to define different Bessel functions for these two values in such a way that the Bessel functions are mostly smooth functions of α.
- The most general solution of Bessel's equation is, $y(x) = AJ_n(x) + BJ_{-n}(x)$, where A and B are arbitrary constants

Glossary

- *Ordinary point*: A point $x = x_0$ is an ordinary point of the second-order linear ordinary differential equation $y^{''} + p(x)y^{'} + q(x) = g(x)$, if $p(x)$, $q(x)$ and $g(x)$ are all analytic at a point x_0.
- *Singular point*: Points that are not ordinary are called singular points of differential equation.
- *Irregular singular point*: Any singular point which is not regular is called irregular singular point.
- *Radius of convergence of the infinite series*: The radius of convergence of the infinite series is the distance to the singularity of the differential equation nearest to the singularity $x = 0$.
- *Frobenius method*: It is named after Ferdinand Georg Frobenius and is a specific technique used to find an infinite series solution for a second order ordinary differential equation.
- *Bessel functions*: These were first defined by the mathematician Daniel Bernoulli and then generalized by Friedrich Bessel are the canonical solutions $y(x)$ of Bessel's differential equation.

Self-assesment questions

1. Find the general solution for the Euler method, $x^2y'' - 3xy' + 4y = 0$

   (a) $\phi(x) = c_1 lnx + c_2x^2$ (b) $\phi(x) = xc_1 lnx^{-2} + c_2 xlnx$

   (c) $\phi(x) = c_1x^2 + c_2x^2 lnx$ (d) $\phi(x) = (c_1 + c_2)lnx$

2. The solution of the differential equation $x^2y'' - 3xy' + 3y = 0$ for $x > 0$ is given by

   (a) $\phi(x) = c_1x^{-1} + c_2x^3$ (b) $\phi(x) = c_1x + c_2x^{-3}$

   (c) $\phi(x) = c_1x + c_2x^3$ (d) $\phi(x) = c_1x^2 + c_2x^{-3}$

3. The indicial polynomial of the equation $x^2y'' + (x^2 - 3x)y' + 3y = 0$ is

   (a) $r^2 = 0$ (b) $r^2 - 3r + 3 = 0$ (c) $r^2 - 4r + 3 = 0$ (d) $r^2 - 4 = 0.$

4. Find the value of the Bessel function of order α of the first kind,

   (a) $J_\alpha(x) = (\frac{x}{2})^{-\alpha} \sum\limits_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-\alpha+1)} (\frac{x}{2})^{2m}$ (b) $J_\alpha(x) = (\frac{x}{2})^{-\alpha} \sum\limits_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)} (\frac{x}{2})^{2m}$

   (c) $J_\alpha(x) = (\frac{x}{2})^{\alpha} \sum\limits_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-\alpha+1)} (\frac{x}{2})^{2m}$ (d) $J_\alpha(x) = (\frac{x}{2})^{\alpha} \sum\limits_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)} (\frac{x}{2})^{2m}$

5. Show that

$$x^{\frac{1}{2}} J_{\frac{1}{2}}(x) = \frac{\sqrt{2}}{\Gamma(\frac{1}{2})} \sin x.$$

6. Show that $K_0'(x) = -K_1(x)$.

EXERCISES

1. Consider the equation

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0,$$

   for $x > 0$.

   (a) Show that there is a solution of the form x^r, where r is a constant.

   (b) Find two linearly independent solutions for $x > 0$, and prove that they are linearly independent.

   (c) Find the two solutions ϕ_1, ϕ_2 satisfying

$$\phi_1(1) = 1, \quad \phi_2(1) = 0,$$
$$\phi_1'(1) = 0, \quad \phi_2'(1) = 1.$$

2. Find two linearly independent solutions of the equation

$$(3x - 1)^2y'' + (9x - 3)y' - 9y = 0$$

   for $x > \frac{1}{3}$.

3. The equation $y' + a(x)y = 0$ has for a solution

$$\phi(x) = \exp\left[- \int_{x_0}^{x} a(t)dt \right].$$

(Here let a be continuous on an interval I containing x_0). This suggests trying to find a solution of

$$L(y) = y'' + a_1(x)y' + a_2(x)y = 0$$

of the form

$$\phi(x) = \exp\left[\int_{x_0}^{x} p(t)dt\right],$$

where p is a function to be determined. Show that ϕ is a solution of $L(y) = 0$ if, and only if, p satisfies the first order non-linear equation

$$y' = -y^2 - a_1(x)y - a_2(x)$$

(Remark: This last equation is called a Riccati equation.)

4. Find all solutions of the following equations for $x > 0$:
   (a) $x^2y'' + 2xy' - 6y = 0$
   (b) $2x^2y'' + xy' - y = 0$
   (c) $x^2y'' + xy' - 4y = x$
   (d) $x^2y'' - 5xy' + 9y = x^3$
   (e) $x^3y''' + 2x^2y'' - xy' + y = 0.$

5. Find all solutions of the following equations for$|x| > 0$:
   (a) $x^2y'' + xy' + 4y = 1$
   (b) $x^2y'' - 3xy' + 5y = 0$
   (c) $x^2y'' - (2+i)xy' + 3iy = 0$
   (d) $x^2y'' + xy' - 4xy = x$.

6. Let ϕ be a solution for $x > 0$ of the Euler equation

$$x^2y'' + axy' + by = 0,$$

   where a, b are constants. Let $\psi(t) = \phi(e^t)$.

   (a) Show that ψ satisfies the equation

$$\psi''(t) + (a-1)\psi(t) + b\psi(t) = 0.$$

   (b) Compute the characteristic polynomial of the equation satisfied by ψ, and compare it with the indicial polynomial of the given Euler equation.
   (c) Show that $\phi(x) = \psi(\log x)$.
   (d) Using $(a), (b), (c)$, and similar facts for $x < 0$ prove Theorem 4.1.

7. Find the singular points of the following equations, and determine those which are regular singular points:
   (a) $x^2y'' + (x + x^2)y' - y = 0$

(b) $3x^2y'' + x^6y' + 2xy = 0$

(c) $x^2y'' - 5y' + 3x^2y = 0$

(d) $xy'' + 4y = 0$

(e) $(1-x^2)y'' - 2xy' + 2y = 0$

(f) $(x^2+x-2)^2y'' + 3(x+2)y' + (x-1)y = 0$

8. Compute the indicial polynomials, and their roots, for the following equations:

   (a) $x^2y'' + (x+x^2)y' - y = 0$

   (b) $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$

9. (a) Show that -1 and 1 are regular singular points for the Legendre equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0.$$

   (b) Find the indicial polynomial, and its roots, corresponding to the point $x = 1$.

10. Find all solutions ϕ of the form

$$\phi(x) = |x|^r \sum_{k=0}^{\infty} c_k x^k, \, (|x| > 0),$$

   for the following equations:

   (a) $3x^2y'' + 5xy' + 3xy = 0$

   (b) $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$. Test each of the series involved for convergence.

11. The equation

$$xy'' + (1-x)y' + \alpha y = 0,$$

   where α is a constant, is called the Laguerre equation.

   (a) Show that this equation has a regular singular point at $x = 0$.

   (b) Compute the indicial polynomial and its roots.

   (c) Find a solution ϕ of the form

$$\phi(x) = x^r \textstyle\sum_{k=0}^{\infty} c_k x^k.$$

   (d) Show that if $\alpha = n$, a non-negative integer, there is a polynomial solution of degree n.

12. Consider the following three equations near $x = 0$:

   (i) $2x^2y'' + (5x + x^2)y' + (x^2 - 2)y = 0$

   (ii) $4x^2y'' - 4xe^xy' + 3(\cos x)y = 0$

   (iii) $(1-x^2)x^2y'' + 3(x+x^2)y' + y = 0$

      (a) Compute the roots r_1, r_2 of the indicial equation for each relative to $x = 0$.

(b) Describe (do not compute) the nature of two linearly independent solutions of each equation near $x = 0$. Using the notation of Theorem 4.4, determine the first non-zero coefficient in $\sigma_2(x)$ if $r_1 = r_2$, and determine whether $c = 0$ in case $r_1 - r_2$, is a positive integer.

13. Consider the equation

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0,$$

where α is a non-negative constant.

(a) Compute the indicial polynomial and its two roots.

(b) Discuss the nature of the solutions near the origin. Consider all cases carefully. Do not compute the solutions.

14. Obtain two linearly independent solutions of the following equations which are valid near $x = 0$:

(a) $x^2y'' + 3xy' + (1 + x)y = 0$

(b) $x^2y'' + 2x^2y' - 2y = 0$

(c) $x^2y'' + 5xy' + (3 - x^3)y = 0$

15. Consider the equation

$$xy' + a(x)y = 0,$$

where

$$a(x) = \sum_{k=0}^{\infty} \alpha_k x^k,$$

and the series converges for $|x| < r_0, r_0 > 0$.

(a) Show formally that there is a solution ϕ of the form

$$\phi(x) = x^r \sum_{k=0}^{\infty} c_k x^k, \quad (c_0 = 1),$$

where $r + \alpha_0 = 0$, and $x > 0$.

(b) Prove that the series obtained converges for $|x| < r_0$.

16. Prove that the series defining J_0 and K_0 converge for $|x| < \infty$.

17. Suppose ϕ is any solution of $x^2y''+xy'+x^2y = 0$ for $x > 0$, and let $\psi(x) = x^{\frac{1}{2}}\phi(x)$, Show that ψ satisfies the equation

$$x^2y'' + (x^2 + \frac{1}{4})y = 0$$

for $x > 0$.

18. Show that J_0 has an infinity of positive zeros. (Hint: If $\psi_0(x) = x^{\frac{1}{2}} J_0(x)$ then ψ_0 satisfies

$$y'' + \left[1 + \frac{1}{4x^2}\right] y = 0, (x > 0)$$

The function χ given by $\chi(x) = \sin x$ satisfies $y'' + y = 0$. Apply Ex. 4 of Sec. 3.4, Chap. 3, to show that there is a zero of J_0 between any two positive zeros of χ.)

19. Show that J_0' satisfies the Bessel equation of order one

$$x^2 y'' + xy' + (x^2 - 1)y = 0.$$

20. (a) Prove that the series defining J_α and $J_{-\alpha}$ converge for $|x| < \infty$.

    (b) Prove that the infinite series involved in the definition of K_a converges $|x| < \infty$.

21. Define $\frac{1}{\Gamma(k)}$ when k is a non-positive integer, to be zero. Show that if n is a positive integer the formula for $J_{-n}(x)$ gives

$$J_{-n}(x) = (-1)^n J_n(x).$$

22. (a) Use the formula for $J_\alpha(x)$ to show that

$$(x^\alpha J_\alpha)'(x) = x^\alpha J_{\alpha-1}(x).$$

    (b) Prove that

$$(x^{-\alpha} J_\alpha)'(x) = -x^\alpha J_{\alpha+1}(x).$$

Answer for check your progress

1. (a) 2. (d) 3. (b) 4. (b) 5. (b) 6. (c)

Suggested Reading

1. M. D. Raisinghania, Advanced Differential Equations, S.Chand and Company Ltd. New Delhi 2001.

2. G. F. Simmons, Differential Equations with Applications and Historical Notes, Tata McGraw Hill, New Delhi, 1974.

3. W. T. Reid, Ordinary Differential Equations, John Wiley and Sons, New York, 1971.

Unit 5

Existence and Uniqueness of Solutions to First Order Equations

OBJECTIVE:
After going through this unit, you will be able to identify the homogeneous differential equations and find the solution of a given differential equation using variables separable. We understand the significance of exact differential equations and the equations reducible to homogeneous form. Finally, we define the significance of successive approximations and the various methods of successive approximations. And also explain how the Lipschitz condition will help to prove the existence and uniqueness theorems.

5.1 Introduction

In this unit we consider the general first order equation

$$y' = f(x, y), \tag{5.1}$$

where f is some continuous function. Only in rather special cases is it possible to find explicit analytic expressions for the solutions of 5.1. We have already considered one such special case; namely, the linear equation

$$y' = g(x)y + h(x), \tag{5.2}$$

where g, h are continuous on some interval I. Any solution ϕ of 5.2 can be written in the form

$$\phi(x) = e^{Q(x)} \int_{x_0}^{x} e^{-Q(t)} h(t) dt + c e^{Q(x)}, \tag{5.3}$$

where

$$Q(x) = \int_{x_0}^{x} g(t) dt,$$

where $x_0 \in I$, and c is a costant. Our main goal is to prove that a wide class of equations of the form 5.1 have solutions, and that solutions to initial value problems are unique. If f is not a linear equation there are certian limitations which must be expected concerning any general existence theorem. To illustrate this consider the equation

$$y' = y^2.$$

Here $f(x, y) = y^2$, and we see f has derivatives of all orders with respect to x and y at every point in the (x, y)-plane. A solution ϕ of this equation satisfying the initial condition

$$\phi(1) = -1$$

is given by

$$\phi(x) = -\frac{1}{x},$$

as can be readily checked. However this solution ceases to exist at $x = 0$, even though f is a nice function there. This example shows that any general existence theorem for 5.1 can only assert the existence of a solution on some interval near-by the initial point.

The above phenomenon does not occur in the case of the linear equation 5.2, for itis clear from 5.3 that any solution ϕ exists on all of the interval I. This points up one of the fundamental difficulties we encounter when we consider nonlinear equations. The equation often gives no clue as to how far a solution will exist.

We prove that initial value problems for equation 5.1 have unique solutions which can be obtained by an approximation process, provided f satisfies an additional condition, the Lipschitz condition. We first concentrate our attention on the case when f is real-valued, and later show how the results carry over to the situation when f is complex-valued.

5.2 Equations with variables separated

A first order equation

$$y' = f(x, y)$$

is said to have the variables separated if f can be written in the form

$$f(x, y) = \frac{g(x)}{h(y)},$$

where g, h are functions of a single argument. In this case we may write our equation as

$$h(y)\frac{dy}{dx} = g(x), \tag{5.4}$$

or

$$h(y)dy = g(x)dx$$

and we readily see the origin of the term "variables separated". For simplicity let us discuss the equation 5.4 in the case g and h are continuous real-valued functions defined for real x and y, respectively. If ϕ is a real-valued solution of 5.4 on some interval I containing a point x_0, then

$$h(\phi(x))\phi'(x) = g(x)$$

for all $x \in I$, and therefore

$$\int_{x_0}^{x} h(\phi(t))\phi'(t)dt = \int_{x_0}^{x} g(t)dt \tag{5.5}$$

for all $x \in I$. Letting $u = \phi(t)$ in the integral on the left in 5.2, we see that 5.2 may be written as

$$\int_{\phi(x_0)}^{\phi(x)} h(u)du = \int_{x_0}^{x} g(t)dt.$$

Conversely, suppose x and y are related by the formula

$$\int_{y_0}^{y} h(u)du = \int_{x_0}^{x} g(t)dt, \tag{5.6}$$

and that this defines implicitly a differentiable function ϕ for $x \in I$. Then this function satisfies

$$\int_{y_0}^{\phi(x)} h(u)du = \int_{x_0}^{x} g(t)dt$$

for all $x \in I$, and differentiating we obtain

$$h(\phi(x))\phi'(x) = g(x),$$

which shows that ϕ is a solution of 5.4 on I. In practice the usual way of dealing with 5.4 is to write it as

$$h(y)dy = g(x)dx$$

(thus separating the variables), and then integrate to obtain

$$\int h(y)dy = \int g(x)dx + c,$$

where c is a constant, and the integrals are anti-derivatives. Thus

$$H(y) = \int h(y)dy, \quad G(x) = \int g(x)dx,$$

represent any two functions H, G such that

$$H' = h, \quad G' = g.$$

Then any differentiable function ϕ which is defined implicitly by the relation

$$H(y) = G(x) + c \tag{5.7}$$

will be a solution of 5.4. Therefore it is usual to identify any solution thus obtained with the relation 5.7. We summarize in the following theorem.

Theorem 5.1 *Let g, h be continuous real-valued functions for $a \leq x \leq b$, $c \leq y \leq d$ respectively, and consider the equation*

$$h(y)y' = g(x). \tag{5.8}$$

If G, H are any functions such that $G' = g, H' = h$, and c is any constant such that the relation

$$H(y) = G(x) + c$$

defines a real-valued differentiable function ϕ for x in some interval I contained in $a \leq x \leq b$, then ϕ will be a solution of 5.8 on I. Conversely, if ϕ is a solution of 5.8 on I, it satisfies the relation

$$H(y) = G(x) + c$$

on I, for some constant c. The simplest example is that case in which $h(y) = 1$. Then $y' = g(x)$, and every solution ϕ has the form

$$\phi(x) = G(x) + c, \tag{5.9}$$

where G is any function on $a \leqq x \leqq b$ such that $G' =, g$, and c is a constant. Moreover, if c is any constant, 5.9 defines a solution of $y' = g(x)$. Thus we have found all solutions of $y' = g(x)$ on $a \leq x \leq b$.

The function ϕ will be a solution of $y' = g(x)/h(y)$ on I, provided $h(\phi(x))$ 0 for all x in I. Another simple case occurs when $g(x) = 1$, for then we have

$$y' = \frac{1}{h(y)}, \tag{5.10}$$

or

$$h(y)dy = dx.$$

Thus, if $H' = h$, any differentiable function defined implicitly by the relation

$$H(y) = x + c, \tag{5.11}$$

where c is a constant, will be a solution of 5.10.

Example 5.1 *Find the solutions of $y' = y^2$.*

Solution: Comparing the given equation with the equation 5.10, we get $h(y) = \dfrac{1}{y^2}$, which is not continuous at $y = 0$. We have

$$\frac{dy}{y^2} = dx,$$

and thus the relation 5.11 becomes

$$-\frac{1}{y} = x + c, \quad \text{or} \quad y = \frac{-1}{x+c}.$$

Thus, if c is any constant, the function ϕ given by

$$\phi(x) = \frac{-1}{x+c} \tag{5.12}$$

is a solution of $y' = y^2$, provided $x \neq -c$.
Note: It is important to remark that the separation of variables method of finding solutions may not yield all solutions of an equation. For example, it is clear from $y' = y^2$ that the function ψ which is identically zero for all x is a solution of $y' = y^2$. However, for no constant c will the ϕ of 5.12 yield this solution.

Example 5.2 *Find the solutions of $y' = 3y^{2/3}$.*

Solution: The given equation can be written as

$$\frac{dy}{y^{2/3}} = 3dx$$

if $y \neq 0$, and hence to

$$y^{1/3} = x + c, \quad \text{or } y = (x+c)^3,$$

where c is a constant. Thus the function ϕ given by

$$\phi(x) = (x+c)^3 \tag{5.13}$$

will be a solution of $y' = 3y^{2/3}$ for any constant c.
Note:

1. The identically zero function is a solution of $y' = 3y^{2/3}$ which can not be obtained from 5.13.

2. The two functions ϕ and ψ given by

   $$\phi(x) = x^3, \quad \psi(x) = 0, \quad (-\infty < x < \infty),$$

   are solutions of $y' = 3y^{2/3}$ which pass through the origin. Actually there are infinitely many functions which are solutions of $y' = 3y^{2/3}$ passing through the origin. To see this let k be any positive number, and define ϕ_k by

   $$\phi_k(x) = 0, \quad (-\infty < x \leq k),$$
   $$\phi_k(x) = (x-k)^2, \quad (k < x < \infty).$$

   Then ϕ_k is a solution of $y' = 3y^{2/3}$ for all real x, and clearly $\phi_k(0) = 0$. This implies that nonlinear equations may have several solutions satisfying a given initial condition.

Let us sum up

1. We have discussed the concept of variable separable method.

2. We have provided the important remark to the separation of variables method with an example.

3. Finally, we rectified some illustrative examples.

Check your progress

1. A function f defined for real x, y is said to be homogeneous of degree k if $f(tx, ty)$ is equal to
   (a) $t^k f(x, y)$ (b) $t^{-k} f(x, y)$ (c) $t^{2k} f(x, y)$ (d) $t^{1/k} f(x, y)$

2. Consider the differential equation $x^2 dy + y(x + y)dx = 0$. Which of the following statements is true.
   (a) The differential equation is linear (b) Variables separable form
   (c) The differential equation is exact (d) None of these

5.3 Exact equations

Suppose the first order equation $y' = f(x, y)$ is written in the form

$$y' = \frac{-M(x, y)}{N(x, y)},$$

or equivalently

$$M(x, y) + N(x, y)y' = 0, \tag{5.14}$$

where M, N are real-valued functions defined for real x, y on some rectangle R. The equation 5.14 is said to be exact in R if there exists a function F having continuous first partial derivatives there such that

$$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N, \tag{5.15}$$

in R. If 5.14 is exact in R, and F is a function satisfying 5.15, then 5.14 becomes

$$\frac{\partial F}{\partial x}(x, y) + \frac{\partial F}{\partial y}(x, y)y' = 0.$$

If ϕ is any solution on some interval I, then

$$\frac{\partial F}{\partial x}(x, \phi(x)) + \frac{\partial F}{\partial y}(x, \phi(x))\phi'(x) = 0, \tag{5.16}$$

for all $x \in I$. If $\Phi(x) = F(x, \phi(x))$, then equation 5.16 just says that $\Phi'(x) = 0$, and hence

$$F(x, \phi(x)) = c,$$

where c is some constant. Thus the solution ϕ must be a function which is given implicitly by the relation

$$F(x, y) = c. \tag{5.17}$$

Looking at this argument in reverse we see that if ϕ is a differentiable function on some interval I defined implicitly by the relation 5.17 then

$$F(x, \phi(x)) = c,$$

for all $x \in I$, and a differentiation yields 5.16. Thus ϕ is a solution of 5.14.

Theorem 5.2 *Suppose the equation*

$$M(x,y) + N(x,y)y' = 0 \tag{5.18}$$

is exact in a rectangle R, and F is a real-valued function such that

$$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N \tag{5.19}$$

in R. Every differentiable function ϕ defined implicitly by a relation

$$F(x,y) = c, \quad (c = \text{ constant })$$

is a solution of 5.14, and every solution of 5.14 whose graph lies in R arises this way.

The problem of solving an exact equation is now reduced to the problem of determining a function F satisfying 5.15. If 5.14 is exact and we write it as

$$M(x,y)dx + N(x,y)dy = \frac{\partial F}{\partial x}(x,y)dx + \frac{\partial F}{\partial y}(x,y)dy = 0$$

we recognize that the left side of this equation is the differential dF of F. This is the explanation of the term "exact"; the left side is an exact differential of a function F. Sometimes an F can be determined by inspection. For example, if the equation

$$y' = -\frac{x}{y} \tag{5.20}$$

is written in the form

$$xdx + ydy = 0$$

it is clear that the left side is the differential of $(x^2 + y^2)/2$. Thus any differentiable function which is defined by the relation

$$x^2 + y^2 = c, \quad (c = \text{ constant })$$

is a solution of 5.20. Note that the equation 5.20 does not make sense when $y = 0$. The above example is also a special case of an equation with variables separated. Indeed any such equation is a special case of an exact equation, for if we write the equation as

$$g(x)dx = h(y)dy$$

it is clear that an F is given by

$$F(x,y) = G(x) - H(y),$$

where $G' = g, H' = h$. How do we recognize when an equation is exact? To see how, suppose

$$M(x,y)dx + N(x,y)dy = 0$$

is exact, and F is a function which has continuous second derivatives such that

$$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N.$$

Then

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial M}{\partial y}, \quad \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

and, since for such a function

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}$$

we must have

$$\frac{\partial M}{\partial y} = \frac{\partial M}{\partial x}.$$

This is the condition we are looking for, since it is true that if this equality is valid, the equation is exact.

Theorem 5.3 *Let M, N be two real-valued functions which have continuous first partial derivatives on some rectangle*

$$R: \quad |x - x_0| \leqq a, \quad |y - y_0| \leqq b.$$

Then the equation

$$M(x, y) + N(x, y)y' = 0$$

is exact in R if, and only if,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{5.21}$$

in R.

Proof:
We have already seen that if the equation is exact, then 5.21 is satisfied. Now suppose 5.21 is satisfied in R. We need to find a function F satisfying

$$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N.$$

To see how to do this, we note that if we had such a function then

$$\begin{aligned} F(x, y) - F(x_0, y_0) &= F(x, y) - F(x_0, y) + F(x_0, y) - F(x_0, y_0) \\ &= \int_{x_0}^{2} \frac{\partial F}{\partial x}(s, y) ds + \int_{s_0}^{y} \frac{\partial F}{\partial y}(x_0, t)\, dt \\ &= \int_{x_0}^{x} M(s, y) ds + \int_{y_0} N(x_0, t)\, dt. \end{aligned}$$

Similarly we would have

$$\begin{aligned} F(x, y) - F(x_0, y_0) &= F(x, y) - F(x, y_0) + F(x, y_0) - F(x_0, y_0) \\ &= \int_{y_0}^{s} \frac{\partial F}{\partial y}(x, t) dt + \int_{x_0}^{z} \frac{\partial F}{\partial x}(s, y_0)\, ds \\ &= \int_{y_0} N(x, t) dt + \int_{x_0}^{z} M(s, y_0)\, ds. \end{aligned} \tag{5.22}$$

We now define F by the formula

$$F(x,y)=\int_{x_0}^{x} M(s,y)ds+\int_{y_0}^{y} N\left(x_0,t\right)dt. \tag{5.23}$$

This definition implies that $F(x_0,y_0)=0$, and that

$$\frac{\partial F}{\partial x}(x,y)=M(x,y),$$

for all (x,y) in R. From 5.22 we would guess that F is also given by

$$F(x,y)=\int_{y_0}^{y} N(x,t)dt+\int_{x_0}^{x} M\left(s,y_0\right)ds. \tag{5.24}$$

This is in fact true, and is a consequence of the assumption 5.21. Once this has been shown, it is clear from 5.24 that

$$\frac{\partial F}{\partial y}(x,y)=N(x,y),$$

for all (x,y) in R, and we have found our F. In order to show that 5.24 is valid, where F is the function given by 5.23, let us consider the difference

$$\begin{aligned}
F(x,y)-&\left[\int_{y_0}^{y} N(x,t)dt+\int_{x_0}^{x} M\left(s,y_0\right)ds\right]\\
&=\int_{x_0}^{x}\left[M(s,y)-M\left(s,y_0\right)\right]ds-\int_{y_0}^{\nu}\left[N(x,t)-N\left(x_0,t\right)\right]dt\\
&=\int_{x_0}^{x}\left[\int_{y_0}^{y}\frac{\partial M}{\partial y}(s,t)dt\right]ds-\int_{y_0}^{\nu}\left[\int_{x_0}^{x}\frac{\partial N}{\partial x}(s,t)ds\right]dt\\
&=\int_{x_0}^{x}\int_{y_0}^{\nu}\left[\frac{\partial M}{\partial y}(s,t)-\frac{\partial N}{\partial x}(s,t)\right]dsdt,
\end{aligned}$$

which is zero by virtue of 5.21.

Example 5.3 *Let us consider the equation*

$$y'=\frac{3x^2-2xy}{x^2-2y} \tag{5.25}$$

which we write as

$$\left(3x^2-2xy\right)dx+\left(2y-x^2\right)dy=0.$$

Here

$$M(x,y)=3x^2-2xy,\quad N(x,y)=2y-x^2,$$

and a computation shows that

$$\frac{\partial M}{\partial y}(x,y)=\frac{\partial N}{\partial x}(x,y)=-2x,$$

which shows that our equation is exact for all x, y. To find an F we could use either of the two formulas 5.23 or 5.24, but the following way is often simpler. We know there is an F such that

$$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N.$$

Thus F satisfies

$$\frac{\partial F}{\partial x}(x, y) = 3x^2 - 2xy,$$

which implies that for each fixed y,

$$F(x, y) = x^3 - x^2 y + f(y), \tag{5.26}$$

where f is independent of x. Now $\partial F/\partial y = N$ tells us that

$$-x^2 + f'(y) = 2y - x^2$$

or that

$$f'(y) = 2y.$$

Thus a choice for f is given by $f(y) = y^2$, and placing this back into 5.26 we obtain finally

$$F(x, y) = x^3 - x^2 y + y^2.$$

Any differentiable function ϕ which is defined implicitly by a relation

$$x^3 - x^2 y + y^2 = c, \tag{5.27}$$

where c is a constant, will be a solution of 5.25, and all solutions of 5.25 arise in this way. Often the solutions are identified with the relations 5.27. It is proved in advanced calculus texts that 5.27 will define a unique differentiable function ϕ near, and passing through, a given point (x_0, y_0) provided that

$$F\left(x_0, y_0\right) = c$$

and that

$$\frac{\partial F}{\partial y}\left(x_0, y_0\right) \neq 0.$$

Notice that the only points (x_0, y_0) satisfying 5.27 for which

$$\frac{\partial F}{\partial y}\left(x_0, y_0\right) = 0$$

are those satisfying

$$-x_0^2 + 2y_0 = 0$$

and these are precisely the points where the given equation 5.25 is not defined. Thus, if (x_0, y_0) is a point for which $(3x^2 - 2xy) / (x^2 - 2y)$ is defined, there will be a unique solution of 5.25 whose graph passes through (x_0, y_0).

Let us sum up

1. We have discussed the significance of exact differential equations and it's solutions.

2. Finally, we solved some illustrative examples.

Check your progress

3. The differential equation $xdy - ydx = 0$ represents
   (a) Parabolas (b) Straight lines (c) Circles (d) None of these

4. For the differential equation $M(x,y) + N(x,y)y' = 0$ to be exact if
   (a) $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$ (b) $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (c) $\frac{\partial^2 M}{\partial^2 x} = \frac{\partial^2 N}{\partial^2 y}$ (d) None of these

5.4 The method of successive approximation

We now face up to the general problem of finding solutions of the equation

$$y' = f(x,y), \tag{5.28}$$

where f is any continuous real-valued function defined on some rectangle

$$R: \quad |x - x_0| \leqq a, \quad |y - y_0| \leqq b, \quad (a, b > 0),$$

in the real (x,y)-plane. Our object is to show that on some interval I containing x_0 there is a solution ϕ of 5.28 satisfying

$$\phi(x_0) = y_0. \tag{5.29}$$

By this we mean there is a real-valued differential function ϕ satisfying 5.29 such that the points $(x, \phi(x))$ are in R for x in I, and

$$\phi'(x) = f(x, \phi(x)),$$

for all x in I. Such a function ϕ is called a solution to the initial value problem

$$y' = f(x,y), \quad y(x_0) = y_0 \tag{5.30}$$

on I. Our first step will be to show that the initial value problem is equivalent to an integral equation, namely

$$y = y_0 + \int_{x_0}^{x} f(t,y)\, dt \tag{5.31}$$

on I. By a solution of this equation on I is meant a real-valued continuous function ϕ on I such that $(x, \phi(x))$ is in R for all x in I, and

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\, dt, \tag{5.32}$$

for all x in I.

Theorem 5.4 *A function ϕ is a solution of the initial value problem 5.30 on an interval I if and only if it is a solution of the integral equation 5.31 on I.*

Proof:
Assume that ϕ is a solution of the initial value problem on

$$\begin{aligned} y' &= f(x,y), \quad y(x_0) = y_0 \\ \Rightarrow \phi'(x) &= f(x,\phi(x)) \quad and \quad \phi(x_0) = y_0 \\ \phi'(t) &= f(t,\phi(t)) \end{aligned} \tag{5.33}$$

on I. Since ϕ is continuous on I, and f is continuous and the rectangle R, the function F defined by

$$F(t) = f(t,\phi(t))$$

is continuous on I. Integrating 5.33 from x_0 to x we obtain

$$\begin{aligned} \int_{x_0}^{x} \phi_t' dt &= \int_{x_0}^{x} f(t,\phi(t))dt \\ [\phi(t)]_{x_0}^{x} &= \int_{x_0}^{x} f(t,\phi(t))dt \\ \phi(x) - \phi(x_0) &= \int_{x_0}^{x} f(t,\phi(t))dt \\ \phi(x) &= \phi(x_0) + \int_{x_0}^{x} f(t,\phi(t))dt \end{aligned}$$

and since $\phi(x_0) = y_0$ we see that ϕ is a solution of 5.31.

$$y = y_0 + \int_{x_0}^{x} f(t,y)dt.$$

Conversely, assume that ϕ is a solution of the integral equation

$$\begin{aligned} y &= y_0 + \int_{x_0}^{x} f(t,y)dt \\ \phi(x) &= \phi(x_0) + \int_{x_0}^{x} f(t,\phi(t))dt. \end{aligned} \tag{5.34}$$

Differentiating with respect to x,using the fundamental theorem of integral calculus, that

$$\begin{aligned} \phi'(x) &= 0 + f(x,\phi(x)) \\ \phi'(x) &= f(x,\phi(x)), \end{aligned}$$

for all x on I. Moreover from 5.34 it is clear that $\phi(x_0) = y_0$, and thus ϕ is a solution of the initial value problem $y' = f(x,y), \quad y(x_0) = y_0$. We now solving 5.31. As a first approximation to a solution we consider the function ϕ_0 defined by

$$\phi(x_0) = y_0.$$

It looks like you are describing a process to solve an initial value problem using successive approximations or iterations. The method you're describing is closely related to the Picard iteration method, which is used to approximate solutions to differential equations.

Given the initial value problem of the form:

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0.$$

The Picard iteration method defines a sequence of function $\phi_n(x)$ that converge to the solution of the initial value problem. Here is how the process works:

- Initial Function: Start with the initial approximation $\phi_0(x) = y_0$.
- Successive approximation : Define the successive approximation by iterating the integral equation:

$$\phi_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_n(t))dt.$$

This process can be summarized as follows:

- Initial approximation:

$$\phi_0(x) = y_0.$$

- First iteration:

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \phi_0(t))dt.$$

  Since $\phi_0(t) = y_0$,this simplifies to:

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)dt.$$

- Second iteration:

$$\phi_2(x) = y_0 + \int_{x_0}^{x} f(t, \phi_1(t))dt.$$

- General iteration:

$$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))dt \quad (k = 0, 1, 2, \cdots). \tag{5.35}$$

Taking the limit of this sequence as $k \to \infty$ gives the solution to the integral equation $\phi_k(x) \to \phi(x)$,

$$\phi(x) = \lim_{k\to\infty} \phi_k(x).$$

Therefore the solution to the original differential equation can be expressed as:

$$\phi(x) = y_0 + + \int_{x_0}^{x} f(t, \phi(t))dt.$$

This limit, if it exists and unique is the solution to the initial value problem $y'(x) = f(x, y(x))$ with the initial condition $y(x_0) = y_0$. Thus ϕ would be our desired solution. We call the functions $\phi_0, \phi_0\phi_0, \cdots$ defined by 5.35 successive approximations to a solution of the integral equation 5.31 or the initial value problem 5.30. One way to picture the successive approximation is to think of a machine S (for solving) which converts functions ϕ into new functions $S(\phi)$ defined by

$$S(\phi)(x) = y_0 + \int_{x_0} x f(t, \phi(t))dt.$$

A solution of the initial value problem 5.30 would then be a function ϕ which moves through the machine untouched, that is, a function satisfying $S(\phi) = \phi$. Starting with $\phi_0(x) = y_0$, we see that S converts ϕ_0 into ϕ_1, and then ϕ_1 into ϕ_2 . In general $S(\phi_k) = \phi_{k+1}$, and ultimately we end up with a ϕ such that $S(\phi) = \phi$.

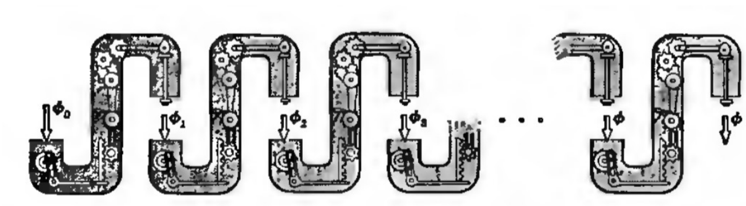


Figure 5.1:

Of course we need to show that the ϕ_k merit the name, that is, we need to show that all the ϕ_k exist on some interval I containing x_0, and that they converge there to a solution of 5.31 or of 5.30. Before doing this let us consider an example

$$y' = xy, \quad y(0) = 1, \tag{5.36}$$

where $x_0 = 0, \quad y_0 = 1$. The integral equation corresponding to this problem is

$$\begin{aligned} y &= y_0 + \int_{x_0}^{x} f(t, y_0)dt \\ y &= 1 + \int_0^x ty\, dt, \end{aligned}$$

and the successive approximation are given by

$$\begin{aligned} \phi(x_0) &= y_0 \\ \Rightarrow \phi(x_0) &= 1, \\ \phi_{k+1}(x) &= 1 + \int_0^x t\phi_k(t)\, dt \quad (k = 0, 1, 2, \cdots). \end{aligned}$$

Thus

$$\begin{aligned} \phi_1(x) &= 1 + \int_0^x t\, dt, \\ &= 1 + \left[\frac{t^2}{2}\right]_0^x = 1 + \frac{x^2}{2}, \\ \phi_2(x) &= 1 + \int_0^x f\left(t, 1 + \frac{t^2}{2}\right)\, dt \end{aligned}$$

$$
\begin{aligned}
&= 1 + \int_0^x t\left(1 + \frac{t^2}{2}\right) dt, \\
&= 1 + \int_0^x t\,dt + \frac{t^3}{2}\,dt, \\
&= 1 + \left[\frac{t^2}{2} + \frac{t^4}{2.4}\right]_0^x, \\
&= 1 + \frac{x^2}{2} + \frac{x^4}{2.4}
\end{aligned}
$$

and it may be established by induction that

$$\phi_k(x) \quad = \quad 1 + \frac{x^2}{2} + \frac{1}{2!}\left(\frac{x^2}{2}\right)^2 + \cdots + \frac{1}{k!}\left(\frac{x^2}{2}\right)^k.$$

We recognize $\phi_k(x)$ as partial sum for the series expansion of the

$$\phi(x) = e^{x^2/2}.$$

We know that this series converges for all real x and this just means that

$$\phi_k(x) \to \phi(x), (k \to \infty),$$

for all real x. The function ϕ is the solution of the problem 5.36. Let us now show that there is an interval I containing x_0 where all the functions $\phi_k,\ k = 0, 1, \cdots$ defined by 5.35 exist. Since f is continuous R, it is bounded there, that is, there exists a constant $M > 0$ such that $|f(x, y)| \leq M$, for all (x, y) in R^*. Let α be the smaller of the two numbers $a, b/M$. Then we prove that the ϕ_k are all defined on $|x - z_0| \leq \alpha$.

Theorem 5.5 *The successive approximations ϕ_k , defined by $\phi(x_0) = y_0$ exist as continuous functions on $I : |x - x_0| \leqq \alpha = \min[a, b/M]$, and (x, ϕ_k) is in R for x in I. Indeed, the ϕ_k satisfy*

$$|\phi_k(x) - y_0| \leqq M|x - x_0|, \tag{5.37}$$

for all x in I.

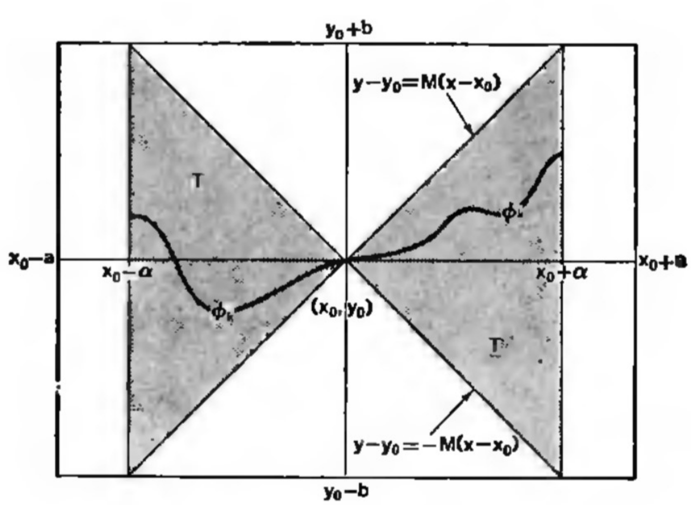


Figure 5.2:

Proof:
We prove this theorem by induction method. We have $\phi(x_0) = y_0$. Clearly ϕ_0 exists on I as a continuous functions, and satisfies 5.37 with $k = 0$. Now $k = 1$,

$$\begin{aligned}\phi_1(x) &= y_0 + \int_{x_0}^{x} f(t, \phi_0(t))dt \\ \phi_1(x) &= y_0 + \int_{x_0}^{x} f(t, y_0)dt. \end{aligned} \tag{5.38}$$

Since f is continuous on R the function $f(t, y_0)$ is continuous on I.

$$\begin{aligned}\phi_1(x) - y_0 &= \int_{x_0}^{x} f(t, y_0)dt \\ |\phi_1(x) - y_0| &\leq \left|\int_{x_0}^{x} f(t, y_0)dt\right| \\ &= \int_{x_0}^{x} |Mdt| = M\int_{x_0}^{x} |dt| \\ &= M|t|_{x_0}^{x} = M|x - x_0| \\ |\phi_1(x) - y_0| &\leq M|x - x_0|\end{aligned}$$

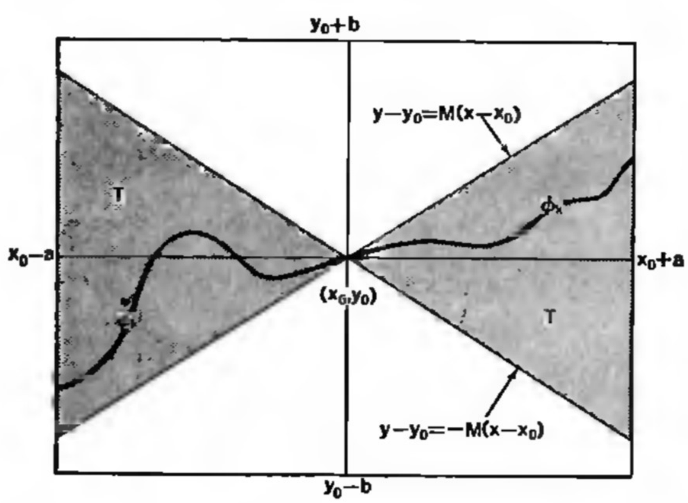


Figure 5.3:

which shows that satisfies ϕ_1 the inequality 5.37. Since f is continuous on R the function F_0 defined by

$$F_0(t) = f(t, y_0)$$

is continuous on I. Thus ϕ_1, which is given by

$$\phi_1(x) = y_0 + \int_{x_0}^{x} F_0(t)dt$$

continuous on I. Now assume the theorem has been proved for the functions $\phi_0, \phi_1, \cdot, \phi_k$. We prove it is valid for ϕ_{k+1}. Indeed the proof is just a repetition of the above. We know that $(t, \phi_k(t))$ is in R for t in I. Thus the function F_k given by

$$F_k(t) = f(t, \phi_k(t))$$

exists for t in I. It is continuous on I since f is continuous on R, and ϕ_k is continuous on I. Therefore ϕ_{k+1}, which is given by

$$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} F_k(t)dt,$$

exists as a continuous function on I. Moreover

$$|\phi_{k+1}(x) - y_0| \leqq \left| \int_{x_0}^{x} |F_k(t)|\, dt \right| \leqq M|x - x_0|,$$

which shows that ϕ_{k+1} satisfies 5.37. The theorem is thus proved by induction.
Note:
Since for x in I, $|x - x_0| \leqq b/M$, the inequality 5.37 implies that

$$|\phi_k(x) - y_0| \leqq b$$

for z in I, which shows that the points $(x, \phi_k(x))$ are in R for x in I.
The precise geometric interpretation of the inequality 5.37 is that the graph of each ϕ_k, lies in the region T in R bounded by the two lines

$$y - y_0 = M(x - x_0), \quad y - y_0 = -M(x - x_0)$$

$$x - x_0 = \alpha, \quad x - x_0 = -\alpha$$

see Figs 5.2 and 5.3.

5.4.1 The Lipschitz condition

Let f be a function defined for (x, y) in a set S. We say f satisfies a Lipschitz condition on S if there exists a constant $K > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq K |y_1 - y_2|,$$

for all $(x, y_1), (x, y_2)$ in S. The constant K is called a Lipschitz constant. If f is continuous and satisfies a Lipschitz condition on the rectangle R, then the successive approximations converge to a solution of the initial value problem on $|x - x_0| \leqq \alpha$. Before we prove this, let us remark that a Lipschitz condition is a rather mild restriction on f.

Theorem 5.6 *Suppose S is either a rectangle*

$$|x - x_0| \leq a, \quad |y - y_0| \leq b, \quad (a, b > 0),$$

or a strip

$$|x - x_0| \leq a, \quad |y| < \infty, \quad (a > 0),$$

and that f is a real-valued function defined on S such that $\partial f/\partial y$ exists, is continuous on S, and

$$\left| \frac{\partial f}{\partial y}(x, y) \right| \leq K, \quad ((x, y) \text{ in } S)$$

for some $K > 0$. Then f satisfies a Lipschitz condition on S with Lipschitz constant K.

Proof:
We have
$$f(x, y_1) - f(x, y_2) = \int_{y_2}^{y_1} \frac{\partial f}{\partial y}(x, t) dt,$$
and hence
$$|f(x, y_1) - f(x, y_2)| \leq \left| \int_{y_2}^{y_1} \right| \frac{\partial f}{\partial y}(x, t) |dt| \leq K |y_1 - y_2|,$$
for all $(x, y_1), (x, y_2)$ in S.
An example of a function satisfying a Lipschitz condition is
$$f(x, y) = xy^2$$
on
$$R: \quad |x| \leq 1, \quad |y| \leq 1.$$
Here
$$\left| \frac{\partial f}{\partial y}(x, y) \right| = |2xy| \leqq 2,$$
for (x, y) on R. This function does not satisfy a Lipschitz condition on the strip
$$S: \quad |x| \leq 1, \quad |y| < \infty,$$
since
$$\left| \frac{f(x, y_1) - f(x, 0)}{y_1 - 0} \right| = |x||y_1|,$$
which tends to infinity as $|y_1| \to \infty$, if $|x| \neq 0$. An example of a continuous function not satisfying a Lipschitz condition on a rectangle is
$$f(x, y) = y^{2/3}$$
on
$$R: |x| \leq 1, \quad |y| \leq 1.$$
Indeed, if $y_1 > 0$,
$$\frac{|f(x, y_1) - f(x, 0)|}{|y_1 - 0|} = \frac{y_1^{2/3}}{y_1} = \frac{1}{y_1^{1/2}},$$
which is unbounded as $y_1 \to 0$.
Let us sum up

1. We have defined the initial function, Initial and successive approximation.
2. We have discussed the significance and various methods of successive approximations.
3. We have characterized the Lipschitz condition.
4. Finally, we solved some suitable examples.

Check your progress

5. If S is a strip $|x - x_0| \leq a$, $|y| < \infty$ (a,0) and if f is real valued continuous function defined on S and $\frac{\partial f}{\partial y}$ exist and also $|\frac{\partial}{\partial y} f(x, y)| \leq K$; $(x, y) \in S$ for a positive constant K.
   (a) f satisfies Lipschitz condition on S with Lipschitz constant K.
   (b) f does not satisfies Lipschitz condition on S with Lipschitz constant K.
   (c) Both (A) and (B) are true.
   (d) None of these.

6. A function satisfying Lipschitz condition is $f(x, y) = xy^2$ on $R : |x| \leq 1, |y| \leq 1$
   (a) 4 (b) 1 (c) 3 (d) 2

5.5 Convergence of the successive approximation

We now prove the main existence theorem.

Theorem 5.7 *(Existence Theorem). Let f be a continuous real-valued functions on the rectangle $R : \quad |x - x_0| \leqq a, \quad |y - y_0| \leqq b \quad (a, b > 0)$, and let $|f(x, y)| \leqq M$, for all (x, y) in R. Further suppose that f satisfies a Lipschitz condition with constant K in R. Then the successive approximations*

$$\phi_0(x) = y_0, \quad \phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))\, dt, \quad (k = 0, 1, 2, \cdots),$$

converge on the internal $I : \quad |x - x_0| \leq a = \min\{a, b/M\}$ to a solution ϕ of the initial value problem $y' = f(x, y), \quad y(x_0) = y_0$ on I.

Proof:
a. Convergence of $\{\phi_k(x)\}$:
The key to the proof is the observation that ϕ_k may be written as

$$\phi_k = \phi_0 + (\phi_1 - \phi_0) + (\phi_2 - \phi_1) + \cdots + (\phi_k - \phi_{k-1}),$$

and, hence $\phi_k(x)$ is a partial sum of the series

$$\phi_k = \phi_0 + \sum_{p=1}^{k} (\phi_p - \phi_{p-1})$$

$$\phi_k(x) = \phi_0(x) + \sum_{p=1}^{\infty} [\phi_p(x) - \phi_{p-1}(x)]\,. \tag{5.39}$$

Therefore to show that the sequence $|\phi_k(x)|$ converges is equivalent to showing that the series 5.39 converges. To prove the latter we must estimate the terms $\phi_p(x) - \phi_{p-1}(x)$ of this series. By Theorem 5.5 the functions ϕ_p all exist as continuous functions on I, and $(x_1, \phi_p(x))$ is in R for x in I. Moreover, as shown in Theorem 5.5,

$$|\phi_1(x) - \phi_0(x)| \leq M|x - x_0| \tag{5.40}$$

for x in I. Writing down the relations defining ϕ_2 and ϕ_1,

$$k = 0, \quad \phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \phi_0(t))\, dt, \tag{5.41}$$

$$k = 1, \quad \phi_2(x) = y_0 + \int_{x_0}^{x} f(t, \phi_1(t))\, dt, \tag{5.42}$$

and subtracting 5.41 from 5.42, we obtain

$$\phi_2(x) - \phi_1(x) = \int_{x_0}^{x} \left[f(t, \phi_1(t)) - f(t, \phi_0(t))\right] dt.$$

Therefore

$$\begin{aligned} |\phi_2(x) - \phi_1(x)| &\leq \left| \int_{x_0}^{x} \left[f(t, \phi_1(t)) - f(t, \phi_0(t))\right] dt \right| \\ &= \int_{x_0}^{x} |f(t, \phi_1(t)) - f(t, \phi_0(t))|\, |dt|, \end{aligned}$$

and since f satisfies the Lipschitz condition

$$|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2|,$$

we have

$$|\phi_2(x) - \phi_1(x)| \leq K \left| \int_{x_0}^{x} |\phi_1(t) - \phi_0(t)| dt \right|.$$

Using 5.40 we obtain

$$|\phi_2(x) - \phi_1(x)| \leq KM \left| \int_{x_0}^{x} |t - x_0| dt \right|.$$

Thus, if $x \geq x_0$,

$$\begin{aligned} |\phi_2(x) - \phi_1(x)| &\leq KM \int_{x_0}^{x} (t - x_0) dt \\ &= KM \left[\left(\frac{t - x_0}{2} \right)^2 \right]_{x_0}^{x} \\ &= KM \left(\frac{x - x_0}{2} \right)^2. \end{aligned} \tag{5.43}$$

In case $x \leq x_0$,

$$\begin{aligned} |\phi_2(x) - \phi_1(x)| &\leq KM \int_{x_0}^{x} (t - x_0) dt \\ &= -KM \int_{x}^{x_0} (t - x_0) dt \\ &= -KM \left[\left(\frac{t - x_0}{2} \right)^2 \right]_{x}^{x_0} \end{aligned}$$

$$= \quad KM\left(\frac{x-x_0}{2}\right)^2. \tag{5.44}$$

We shall prove by induction that

$$|\phi_p(x)-\phi_{p-1}(x)| \leqq \frac{MK^{p-1}|x-x_0|^p}{p!}, \tag{5.45}$$

for all x in I. We have seen that this is true for $p=1$ and $p=2$. Let us assume $x \geq x_0$; the proof is similar for $x \leq x_0$. Assume 5.45 for $p=m$. Using the definition of ϕ_{m+1} and ϕ_m, we obtain

$$\begin{aligned}
\phi_{m+1}(x) &= y_0 + \int_{x_0}^{x} f(t,\phi_m(t))\,dt, \\
\phi_m(x) &= y_0 + \int_{x_0}^{x} f(t,\phi_{m-1}(t))\,dt, \\
\phi_{m+1}(x)-\phi_m(x) &= \int_{x_0}^{x} [f(t,\phi_m(t)) - f(t,\phi_{m-1}(t))]\,dt,
\end{aligned} \tag{5.46}$$

and thus

$$\begin{aligned}
|\phi_{m+1}(x)-\phi_m(x)| &\leq \left|\int_{x_0}^{x} [f(t,\phi_m(t)) - f(t,\phi_{m-1}(t))]\,dt\right|, \\
&= \int_{x_0}^{x} |f(t,\phi_m(t)) - f(t,\phi_{m-1}(t))|\,dt.
\end{aligned}$$

Using the Lipschitz condition we get

$$\begin{aligned}
|\phi_{m+1}(x)-\phi_m(x)| &\leq K\left|\int_{x_0}^{x} |\phi_m(t)-\phi_{m-1}(t)|dt\right|. \\
&= K\int_{x_0}^{x} |\phi_m(t)-\phi_{m-1}(t)|dt. \\
&= \frac{MK^{m-1}|x-x_0|^m}{m!} \\
&= MK^m\frac{|x-x_0|^{m+1}}{(m+1)!}.
\end{aligned}$$

This is just 5.45 for $p=m+1$, and hence 5.45 is valid for all $p=1,2,\cdots$, by induction. It follows from 5.45 that the infinite series

$$\phi_k(x) \leq \phi_0(x) + \sum_{p=1}^{\infty} [\phi_p(x)-\phi_{p-1}(x)] \tag{5.47}$$

is absolutely convergent on I, that is, the series

$$|\phi_0(x)| + \sum_{p=1}^{\infty} |\phi_p(x)-\phi_{p-1}(x)| \tag{5.48}$$

is convergent on I. Indeed, from 5.45 we see that

$$\begin{aligned}
\sum_{p=1}^{\infty} |\phi_p(x) - \phi_{p-1}(x)| &\le \sum_{p=1}^{\infty} \frac{MK^{p-1}|x-x_0|^p}{p!} \\
&= \frac{M}{K} \sum_{p=1}^{\infty} \frac{K^p|x-x_0|^p}{p!} \\
&= \frac{M}{K} e^{K|x-x_0|}
\end{aligned}$$

which shows that the p-th term of the series in 5.48 is less than or equal to M/K times the p-th term of the power series for $e^{K|x-x_0|}$. Since the power series for $e^{K|x-x_0|}$ is convergent, the series 5.48 is convergent for x in I. This implies that the series 5.39 is convergent on I. Therefore the k-th partial sum of 5.39, which is just $\phi_k(x)$ tends to a limit $\phi(x)$ as $k \to \infty$, for each x in I.

b. Properties of the limit ϕ:

This limit function ϕ is a solution to our problem on I. First, let us show that ϕ is continuous on I. This may be seen in the following way. If x_1, x_2 are in I

$$\begin{aligned}
\phi_{k+1}(x_1) &= y_0 + \int_{x_0}^{x_1} f(t, \phi_k(t))\, dt, \\
\phi_{k+1}(x_2) &= y_0 + \int_{x_0}^{x_2} f(t, \phi_k(t))\, dt, \\
|\phi_{k+1}(x_1) - \phi_{k+1}(x_2)| &\le \left| \int_{x_2}^{x_1} f(t, \phi_x(t))\, dt \right|, \\
&= \int_{x_2}^{x_1} |f(t, \phi_x(t))|\, |dt|, \\
&= M \int_{x_2}^{x_1} |dt|, \\
&= M|x_1 - x_2|,
\end{aligned}$$

which implies, by letting $k \to \infty$,

$$|\phi(x_1) - \phi(x_2)| \le M|x_1 - x_2|. \tag{5.49}$$

This shows that as $x_2 \to x_1$, $\phi(x_2) \to \phi(x_1)$, that is, ϕ is continuous on I. Also, letting $x_1 = x, x_2 = x_0$ in 5.49 we obtain

$$|\phi(x) - y_0| \le M|x - x_0|,$$

which implies that the point $(x, \phi(x))$ are in R for all x in I.

c. Estimate for $|\phi(x) - \phi_k(x)|$:

We now estimate $|\phi(x) - \phi_k(x)|$. We have

$$\phi(x) = \phi_0(x) + \sum_{p=1}^{\infty} [\phi_p(x) - \phi_{p-1}(x)],$$

$$\phi_k(x) = \phi_0(x) + \sum_{p=1}^{k} \left[\phi_p(x) - \phi_{p-1}(x)\right].$$

Therefore, using 5.45, we find that

$$\begin{aligned}
|\phi(x) - \phi_k(x)| &= \left|\sum_{p=1}^{\infty} [\phi_p(x) - \phi_{p-1}(x)] - \sum_{p=1}^{k} [\phi_p(x) - \phi_{p-1}(x)]\right| \\
&\leq \left|\sum_{p=k+1}^{\infty} (\phi_p(x) - \phi_{p-1}(x))\right| \\
&= \sum_{p=k+1}^{\infty} |\phi_p(x) - \phi_{p-1}(x)|, \\
&= \sum_{p=k+1}^{\infty} \frac{MK^P|x-x_0|^P}{KP!} \\
&= \sum_{p=k+1}^{\infty} \frac{M}{K}\frac{K^P|x-x_0|^P}{P!} \\
&= \frac{M}{K}\sum_{p=k+1}^{\infty} \frac{K^P\alpha^P}{P!} \\
&= \frac{M}{K}\sum_{p=k+1}^{\infty} \frac{(K\alpha)^P}{P!} \\
&= \frac{M}{K}\left[\frac{(K\alpha)^{k+1}}{(k+1)!} + \frac{(K\alpha)^{k+2}}{(k+2)!} + \cdots\right] \\
&= \frac{M}{K}\frac{(K\alpha)^{k+1}}{(k+1)!}\left[1 + \frac{K\alpha}{(k+2)} + \frac{(K\alpha)^2}{(k+2)(k+3)} + \cdots\right] \\
&= \frac{M}{K}\frac{(K\alpha)^{k+1}}{(k+1)!}\sum_{p=0}^{\infty}\frac{(K\alpha)^p}{P!} \quad k = 0, 1, 2, \cdots \\
&< \frac{M}{K}\frac{(K\alpha)^{k+1}}{(k+1)!}e^{K\alpha}. \qquad (5.50)
\end{aligned}$$

Letting $\epsilon_k = \frac{(K\alpha)^{k+1}}{(k+1)!}$, we see that $\epsilon_k \to 0$ as $k \to \infty$, since ϵ_k is a general term for the series for $e^{K\alpha}$. In terms of ϵ_k may be written as

$$|\phi(x) - \phi_k(x)| \leqq \frac{M}{K}e^{K\alpha}\epsilon_k, \qquad (5.51)$$

where $\epsilon_k = \frac{(K\alpha)^{k+1}}{(k+1)!} \to 0, \;\; |\phi(x) - \phi_k(x)|$ as $k \to \infty$

d. The limit ϕ is a solution:

To complete the proof we must show that $\phi(x)$ is a function of the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ that is, we have to prove that

$$\phi(x) \;\; = \;\; y_0 + \int_{x_0}^{x} f(t, \phi(t))dt \qquad (5.52)$$

$$\phi(x) - \phi(x_0) = \int_{x_0}^{x} f(t, \phi(t))dt,$$

for all x in I. The right side of 5.52 makes sense for ϕ is continuous on I, f is continuous on R, and thus the function F is given by

$$F(t) = f(t, \phi(t))dt,$$

is continuous on I. Now

$$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))dt$$

and $\phi_{k+1}(x) \to \phi(x)$, as $k \to \infty$. Thus to prove 5.52 we must show that for each x in I

$$\int_{x_0}^{x} f(t, \phi_k(t))dt \to \int_{x_0}^{x} f(t, \phi(t))dt \quad (k \to \infty) \tag{5.53}$$

we have

$$\begin{aligned} |\phi(x) - \phi_{k+1}(x)| &= \left| \int_{x_0}^{x} f(t, \phi(t))dt - \int_{x_0}^{x} f(t, \phi_k(t))dt \right| \\ &\leqq \int_{x_0}^{x} |f(t, \phi(t)) - f(t, \phi_k(t))|dt \\ &= k \int_{x_0}^{x} |\phi(t) - \phi_k(t)|dt \end{aligned} \tag{5.54}$$

using the fact that f satisfies a Lipschitz condition. The estimate 5.51 can now be used in 5.52 to obtain

$$\begin{aligned} \left| \int_{x_0}^{x} f(t, \phi(t))dt - \int_{x_0}^{x} f(t, \phi_k(t))dt \right| &\leqq k \int_{x_0}^{x} \left| \frac{M}{K} \epsilon_k e^{K\alpha} \right| dt \\ &= M \epsilon_k e^{K\alpha} \int_{x_0}^{x} |dt| \\ &= M \epsilon_k e^{K\alpha} (|x - x_0|) \to 0, \end{aligned}$$

which tends to zero as $k \to \infty$, for each x in I. This proves 5.53 hence that ϕ satisfies 5.52. Thus our proof of this theorem is now complete.

Theorem 5.8 *The k-th successive approximation ϕ_k to the solution ϕ of the initial value problem of the above theorem satisfies*

$$|\phi(x) - \phi_k(x)| \leqq \frac{M}{K} \frac{(K\alpha)^{k+1}}{(k+1)!} e^{K\alpha},$$

for all x in I.

Let us sum up

1. We have discussed the existence for convergence of the successive approximation.

2. We have proved the properties of limit function and $k-$th successive approximation.

Check your progress

7. The k-th successive approximation ϕ_k to the solution ϕ of the BVP satisfies
   (a) $|\phi(x) - \phi_k(x)| \leq \frac{M}{K}\frac{(K\alpha)^{k+1}}{(k+1)!}e^{K\alpha}$ (b) $|\phi(x) - \phi_k(x)| \leq \frac{M}{K}\frac{(K\alpha)^{k+1}}{k!}e^{K\alpha}$
   (c) $|\phi(x) - \phi_k(x)| \leq \frac{M}{K}\frac{(K\alpha)^{k+1}}{(k+1)!}e^{K\alpha^2}$ (d) $|\phi(x) - \phi_k(x)| \leq \frac{M}{K}\frac{(K\alpha)^{k+1}}{k!}e^{K\alpha^2}$

8. state the existence theorem for successive approximation.

Summary

The focus of this unit is the theoretical aspect of ODEs, particularly the conditions under which solutions exist and are unique. Topics discussed include:

- In-depth discussion of the conditions under which unique solutions to ODEs exist (Picard's theorem).

- An exact equation is a type of differential equation that can be solved by finding a function whose total derivative matches the given equation. It is typically written in the form:
$$M(x,y)dx + N(x,y)dy = 0$$

- The method of successive approximation (also known as Picard's method) is an iterative technique for solving differential equations by repeatedly refining an initial guess for the solution. Starting from an initial estimate, the method generates a sequence of functions that converge to the exact solution of the differential equation.

- The Lipschitz condition ensures that a function $f(x,y)$ satiesfies $|f(x_1)+f(x_2)| \leq K|x_1 + X_2|$ for a constant K. It guarantees that the function's behavior is not too erratic, which is useful for ensuring the existence and uniqueness of solutions to differential equations.

- The convergence of successive approximation ensures that if a differential equation satisfies the Lipschitz condition, then the sequence of approximations generated by the method will converge to the true solution. This convergence relies on the iterative process refining the solution with each step based on the previous approximation.

Glossary

- *Variable separable*: A differential equation is said to be separable if the variables can be separated. To solve the equation it is integrated on both sides, i.e first separating the variables and then integrating.

- *Exact differential equation*: A differential equation is said to be exact if it can be obtained from its primitive equation directly by differentiation and without involving any further process of reduction, elimination, multiplication, etc.

- *Picard's iteration method*: A technique for finding approximate solutions to differential equations by iteratively refining an initial guess. Each iteration involves substituting the current approximation into the differential equation to generate a new, hopefully more accurate, approximation.

- *Iteration method*: A numerical technique for finding approximate solutions by repeatedly refining an initial guess using a defined iterative process.

- *Lipschitz condition*: The Lipschitz condition requires that the difference in function values is bounded by a constant times the difference in input values. This ensures the function does not vary too quickly and helps in guaranteeing the existence and uniqueness of solutions to differential equations.

Self-assesment questions

1. If the IVP satisfies Lipschitz condition, then it must have
   (a) Only one solution (b) Infinite number of solution
   (c) Unique solution (d) None of these

2. A function satisfying Lipschitz condition is $f(x, y) = xy^2$ on $R : |x| \leq 1, |y| \leq 1$
   (a) 4 (b) 1 (c) 3 (d) 2

3. For the IVP, $\frac{dy}{dx} = y^2 + cos^2(x), x > 0$. The largest interval of existence of the solution predicted by Picard's theorem is,
   (a) $[0, 1]$ (b) $[0, \frac{1}{2}]$ (c) $[0, \frac{1}{3}]$ (d) $[0, \frac{1}{4}]$

4. Find out the function which does not satisfies a Lipschitz condition on rectangle $R : |x| \leq 1, |y| \leq 1$?
   (a) $f(x, y) = xy^{\frac{1}{2}}$ (b) $f(x, y) = xy^2$ (c) $f(x, y) = x^2y^2$ (d) $f(x, y) = y^{\frac{2}{3}}$

5. Consider the differential equation $(xy + x^2) + (y^2 - y)y' = 0$. Which of the following statements is true.
   (a) The differential equation is linear (b) Variables separable form
   (c) The differential Equation is exact (d) None of these

6. The integrating factor of differential equation $x^2dy + y(x + y)dx = 0$, is
   (a) $\dfrac{1}{x}$ (b) $\dfrac{1}{x^2}$ (c) $\dfrac{1}{x^3}$ (d) $\dfrac{1}{x^4}$

7. Find out the differential equation $\frac{dy}{dx} - x\tan(y - x) = 1$, is
   (a) homogeneous (b) variable seperable (c) linear (d) exact

8. Find out the differential equation $(x + 2y)(dx - dy) = dx + dy$, is
   (a) linear (b) variable seperable (c) homogeneous (d) exact

9. One of the integrating factor of differential equation $(y^2 - 3xy)dx + (x^2 - xy)dy = 0$, is
   (a) $\dfrac{1}{x^2y^2}$ (b) $\dfrac{-1}{2x^2y}$ (c) $\dfrac{1}{xy^2}$ (d) $\dfrac{1}{xy}$

10. Let $y : \mathbb{R} \to \mathbb{R}$ be differentiable satisfying the differential equations, $\frac{dy}{dx} = f(y)$, $x \in \mathbb{R}; y(0) = y(1) = 0$, where $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function. Then,

(a) $y(x) = 0$ iff $x \in 0, 1$ (b) y is bounded

(c) y is strictly increasing (d) y' is bounded.

EXERCISES

1. Find all real-valued solutions of the following equations:

   (a) $y' = x^2 y$

   (b) $yy' = x$

   (c) $y' = \dfrac{x + x^2}{y - y^2}$

   (d) $y' = \dfrac{e^{x-y}}{1 + e^x}$

   (e) $y' = x^2y^2 - 4x^2$.

2. (a) Show that the solution ϕ of

   $$y' = y^2$$

   which passes through the point (x_0, y_0) is given by

   $$\phi(x) = \frac{y_0}{1 - y_0(x - x_0)}.$$

   (Note: The identically zero solution can be obtained from this formula by letting $y_0 = 0$.)

   (b) For which x is ϕ a well-defined function?

   (c) For which x is ϕ a solution of the problem

   $$y' = y^2, \quad y(x_0) = y_0?$$

3. (a) Find the solution of $y' = 2y^{\frac{1}{2}}$ passing through the point (x_0, y_0), where $y_0 > 0$.

   (b) Find all solutions of this equation passing through $(x_0, 0)$.

4. (a) (a) Show that the method of Ex. 5 can be used to reduce an equation of the form

   $$y' = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right)$$

   to a homogeneous equation.

   (b) Solve the equation

   $$y' = \frac{1}{2}\left(\frac{x + y - 1}{x + 2}\right)^2.$$

5. The equations below are written in the form $M(x,y)dx + N(x,y)dy = 0$, where M, N exist on the whole plane. Determine which equations are exact there, and solve these.

   (a) $2xydx + (x^2 + 3y^2)dy = 0$
   (b) $(x^2 + xy)dx + xydy = 0$
   (c) $e^x dx + (e^y(y + 1))dy = 0$
   (d) $\cos x \cos^2 ydx - \sin x \sin 2ydy = 0$
   (e) $x^2y^3dx - x^2y^2dy = 0$
   (f) $(x + y)dx + (x - y)dy = 0$
   (g) $(2ye^{2x} + 2x \cos y)dx + (e^{2x} - x^2 \sin y)dy = 0$
   (h) $(3x^2 \log|x| + x^2 + y)dx + xdy = 0$.

6. Even though an equation $M(x,y)dx+N(x,y)dy = 0$ may not be exact, sometimes it is not too difficult to find a function u, nowhere zero, such that,

$$u(x,y)M(x,y)dx + u(x,y)N(x,y)dy = 0$$

   is exact. Such a function is called an integrating factor. For example,

$$ydx - xdy = 0$$

   is not exact, but multiplying the equation by $u(x,y) = \frac{1}{y^2}$ makes it exact for $y \neq 0$. Solutions are then given by $y = cx$. Find an integrating factor for each of the following equations, and solve them.

   (a) $(2y^3 + 2)dx + 3xy^2dy = 0$
   (b) $\cos x \cos ydx - 2 \sin x \sin ydy = 0$
   (c) $(5x^3y^2 + 2y)dx + (3x^4y + 2x)dy = 0$
   (d) $(e^y + xe^y)dx + xe^ydy = 0$
   (Note: If you have trouble discovering integrating factors, do Exs. 3-5 first.)

7. (a) Under the same conditions as in Ex. 3, show that if

$$M(x,y)dx + N(x,y)dy = 0,$$

   has an integrating factor u, which is a function of y alone, then

$$q = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

   is a continuous function of y alone.

   (b) If q is continuous, and independent of x, show that an integrating factor is given by

$$u(y) = e^{Q(y)},$$

   where Q is any function such that $Q' = q$.

8. Consider the linear equation of the first order

$$y' + a(x)y = b(x),$$

where a, b are continuous on some interval I.

(a) Show that there is an integrating factor which is a function of x alone. (Hint: Ex. 4.)

(b) Solve this equation, using an integrating factor.

9. Consider the initial value problem

$$y' = 3y + 1, \quad y(0) = 2.$$

(a) Show that all the successive approximations $\phi_0, \phi_1, \cdots$ exist for all real x.

(b) Compute the first four approximations $\phi_0, \phi_1, \phi_2, \phi_3$ to the solution.

(c) Compute the solution.

(d) Compare the results of (b) and (c).

10. For each of the following problems compute the first four successive approximations $\phi_0, \phi_1, \phi_2, \phi_3$:

(a) $y' - x^2 + y^2, \;\; y(0) = 0$

(b) $y' = 1 + xy, \;\; y(0) = 1$

(c) $y' = y^2, \;\; y(0) = 0$

(d) $y' = y^2, \;\; y(0) = 1.$

11. Consider the problem

$$y' = x^2 + y^2, \;\; y(0) = 0,$$

on

$$R : |x| \leq 1, |y| \leq 1.$$

(a) Compute an upper bound M for the function $f(x, y) = x^2 + y^2$ on R.

(b) On what interval containing $x = 0$ will all the successive approximations exist, and be such that their graphs are in R?

12. By computing appropriate Lipschitz constants, show that the following functions satisfy Lipschits conditions on the sets S indicated:

(a) $f(x, y) = 4x^2 + y^2$, on $S: \;\; |x| \leq 1, \;\; |y| \leq 1.$

(b) $f(x, y) = x^2 \cos^2 y + y \sin^2 x$ on $S : |x| \leq 1, \;\; |y| < \infty.$

(c) $f(x, y) = x^3 e^{-xy^2}$, on $S: \;\; 0 \leq x \leq a, \;\; |y| < \infty, \;\; (a > 0)$

(d) $f(x,y) = a(x)y^2 + b(x)y + c(x)$, on $S: \ |x| \leq 1, \ |y| \leq 2$,
$(a, \ b, \ c$ are continuous functions on $|x| \leq 1)$

(e) $f(x,y) = a(x)y + b(x)$, on $S: \ |x| \leq 1, \ |y| < \infty$,
$(a, \ b$ are continuous functions on $|x| \leq 1)$

13. (a) Show that the function f given by

$$f(x,y) = y^{\frac{1}{2}}$$

does not satisfy a Lipschits condition on

$$R: \ |x| \leq 1, \ 0 \leq y \leq 1.$$

(b) Show that this f satisfies a Lipschits condition on any rectangle R of the form

$$R: \ |x| \leq a, \ b \leq y \leq c, \ (a, \ b, \ c \ > \ 0).$$

14. (a) Show that the function f given by

$$f(x,y) = x^2|y|$$

satisfies a Lipschitz condition on

$$R: \ |x| \leq 1, \ |y| \leq 1.$$

(b) Show that $\frac{\partial f}{\partial y}$ does not exist at $(x,0)$ if $x \neq 0$.

15. Consider the problem

$$y' = 1 - 2xy, y(0) = 0.$$

(a) Since the differential equation is linear, an expression can be found for the solution. Find it.

(b) Consider the above problem on R:

$$R: |x| \leq \tfrac{1}{2}, \ |y| \leq 1.$$

If $f(x,y) = 1 - 2xy$, show that

$$|f(x,y)| \leq 2, \ \ ((z,y) \in \text{R}),$$

and that all the successive approximations to the solution exist on $|x| \leq \frac{1}{2}$ and their graphs remain in R.

(c) Show that f satisfies a Lipschitz condition on R, with Lipschitz constant $K = 1$, and therefore by Theorem 5.7 the successive approximations converge to a solution ϕ of the initial value problem on $|x| \leq \frac{1}{3}$.

(d) Show that the approximation a satisfies

$$|\phi(x) - \phi_3(x)| < 0.01$$

for $|x| \leq \frac{1}{2}$.

(e) Compute ϕ_3.

16. Consider the problem

$$y' = 1 + y^2, \quad y(0) = 0.$$

   (a) Using separation of variables, find the solution ϕ of this problem. (It is not difficult to convince oneself that the separation of variables technique gives the only solution of the problem.) On what interval does ϕ exist?

   (b) Show that all the successive approximations $\phi_0, \phi_1, \phi_2 \cdots$ exist for all real x.

   (c) Show that $\phi_k \to \phi(x)$ for each x satisfying $|x| \leq \frac{1}{2}$. (Hint: Consider $f(x,y) = 1 + y^2$ on

$$R: \ |x| \leq \tfrac{1}{2}, \ |y| \leq 1.$$

   Show that $\alpha = \frac{1}{2}$.)

Answers for check your progess

1. (a) 2. (b) 3. (b) 4. (b) 5. (a) 6. (d) 7. (a)

8. Existence Theorem: Let f be a continuous real-valued functions on the rectangle $R: \quad |x - x_0| \leqq a, \quad |y - y_0| \leqq b \quad (a, b > 0)$, and let $|f(x,y)| \leqq M$, for all (x,y) in R. Further suppose that f satisfies a Lipschitz condition with constant K in R. Then the successive approximations

$$\phi_0(x) = y_0, \quad \phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))\, dt, \quad (k = 0, 1, 2, \cdots),$$

converge on the internal $I: \quad |x - x_0| \leq a = \min\{a, b/M\}$ to a solution ϕ of the initial value problem $y' = f(x,y), \quad y(x_0) = y_0$ on I.

Suggested readings

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